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New counterexamples to A. D. Alexandrov's hypothesis

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Abstract. The paper presents a series of principally different C^{∞}-smooth counterexamples to the following hypothesis on a characterization of the sphere: Let $K \subset \mathbb{R}^3$ be a smooth convex body. If at every point of ∂K, we have $R_1 \leqslant C \leqslant R_2$ for a constant C, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K.)

The hypothesis was proved by A. D. Alexandrov and H. F. Münzner for analytic bodies. For the case of general smoothness it has been an open problem for years. Recently, Y. Martinez-Maure has presented a C^2-smooth counterexample to the hypothesis.

Key words. Virtual polytope, fan, hérisson, hyperbolic hérisson, saddle surface.

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1 Introduction

The present paper aims at a construction of a series of counterexamples to the following old hypothesis.

Let $K \subset \mathbb{R}^3$ be a smooth convex body. Suppose the inequality $R_1 \leqslant C \leqslant R_2$ is valid at every point of ∂K for a constant C. Then K is a ball of radius C. ($R_1 \leqslant R_2$ are the principal curvature radii of ∂K.)

This hypothesis was proved by A. D. Alexandrov and H. F. Münzner for analytic bodies (see [1] and [14]). There are also some partial results when the hypothesis is valid for nonanalytic surfaces (see [6], [15], and [24]). For the general case of smoothness, the hypothesis has remained an open problem for years. For a long time, mathematicians were certain that it is also true for all smooth convex bodies, but gradually abandoned their attempts to prove it. Only recently, nearly at one and the same time (and absolutely independently) two mutually contradicting papers have appeared: one by A. V. Pogorelov containing an erroneous proof of the hypothesis for C^2-surfaces (see [19] and [20]) and another one by Y. Martinez-Maure [10] containing a C^2-smooth counterexample to the hypothesis. Despite their independence, both authors used the same trick (which also can be traced in the early paper of Alexandrov [1]): they reduced the problem to the consideration of smooth hyperbolic hérissons, that is, a special type of saddle surfaces. (The theory of hérissons is widely developed in [7], [9], [10], [11], [22], and other papers. Although Alexandrov and

Pogorelov used a different terminology, their constructions are equivalent to those in the theory of hérissons.)

This trick reveals a simple relationship between the hypothesis and the extrinsic geometry of saddle surfaces. It is remarkable that the latter subject was investigated rather actively in the 60s of the last century by some Russian mathematicians, but without any connection with the hypothesis (see [4] and [23] for detailed reviews).

Pogorelov's paper contains an uncorrectable mistake. The example of Martinez-Maure is correct.

On the other hand, the investigation of hérissons (which are, roughly speaking, Minkowski differences of smooth convex bodies) has an obvious interplay with the theory of virtual polytopes (which are, roughly speaking, Minkowski differences of convex polytopes). Virtual polytopes were introduced originally by A. Pukhlikov and A. Khovanskij [5] for some reasons of algebraic geometry. They also appeared in the polytope algebras of P. McMullen. In addition, recently Y. Martinez-Maure has also presented a general theory of virtual polytopes (he calls them "discrete hérissons"), which leads to the same notion [11].

The counterexample of Martinez-Maure is based on a construction of a C^2-smooth saddle surface containing 4 cross-caps (see Figure 2.4). The surface is given by an explicit formula. Later, Y. Martinez-Maure presented a polytopal version of his counterexample, namely, a discrete hérisson (i.e., a virtual polytope) which has 4 discrete cross-caps (see Figure 5.4).

In the present paper, at an attempt to obtain new counterexamples, we move in the opposite direction: we start (Section 5) by constructing a hyperbolic virtual polytope with N cross-caps ($N \geqslant 4$ is any even number) and after that, using smoothening techniques, we obtain (Theorem 6.1) a C^∞-smooth hyperbolic hérisson with N smooth cross-caps. The latter gives us the desired series of counterexamples.

The paper is organized as follows. Section 2 contains all necessary notions concerning smooth hérissons and explains the above-mentioned trick of Martinez-Maure and Pogorelov. Section 3 recalls briefly the notion of virtual polytopes. Section 4 introduces the notions of hyperbolic hérissons and hyperbolic virtual polytopes. Section 5 presents a construction of a hyperbolic virtual polytope with N cross-caps ($N = 4, 6, 8, \ldots$). Section 6 describes a smooth hyperbolic approximation of such a polytope. This yields the desired series of counterexamples.

2 Smooth hérissons and the hypothesis

The present section reviews some methods and constructions of [7], [9], [10], [11], [20], and some other papers concerning hérissons in $\mathbb{R}^3$.

As is known, the C^2-smooth convex bodies form a semigroup $\mathscr{H}$ with respect to the Minkowski addition $\otimes$. Since the usual cancellation law is valid in this semigroup, its Grothendieck group $\mathscr{H}^*$ coincides with the group of formal expressions of type $B_1 \otimes B_2^{-1}$, where $B_1, B_2 \in \mathscr{H}$.

Remark 2.1. In the polytope algebra (see [5], [12], and [13]), the Minkowski addition plays the role of multiplication. Moreover, it reflects the multiplication in Picard groups of most of toric varieties. For this reason, we use $\otimes$ instead of more usual $+$. By $(\cdot)^{-1}$ we mean inversion with respect to $\otimes$.

Owing to linearity, there exist reasonable definitions of support function, support planes and principal curvature radii for elements of $\mathscr{H}^*$.

Definition 2.2. Let $B \in \mathscr{H}^*$ and $B = B_1 \otimes B_2^{-1}$, where B_1 and B_2 are convex smooth bodies. The *support function* h *of* B is defined as the (pointwise) difference of the support functions of B_1 and B_2:

$$h = h_{B_1} - h_{B_2}.$$

For each point ξ lying on the unit sphere S^2 centered at the origin O (which is identified with the set of unit vectors in $\mathbb{R}^3$), we define the *oriented support plane* $e_B(\xi)$ *with the normal vector* ξ of B by the equation

$$(x, \xi) = h(\xi).$$

By the *hérisson* B we mean the envelope of the family of planes $\{e_B(\xi)\}_{\xi \in S^2}$. It is a sphere-homeomorphic surface with possible self-intersections and self-overlappings. We say that a hérisson B is C^2-smooth (C^∞-smooth) if its support function is C^2-smooth (C^∞-smooth).

As a set of points, a hérisson B coincides with the image of the mapping

$$\varphi : S^2 \to \mathbb{R}^3, \quad (x, y, z) \to (h'_x(x, y, z), h'_y(x, y, z), h'_z(x, y, z)).$$

Definition 2.3. Analogously to the classical convex case, we define the *principal curvature radii* R_1 and R_2 of a hérisson B at the point $\xi \in S^2$ (or at the point $\varphi(\xi) \in B$) as the eigenvalues of the matrix

$$\begin{pmatrix} h''_{xx}(\xi) & h''_{xy}(\xi) \\ h''_{yx}(\xi) & h''_{yy}(\xi) \end{pmatrix}.$$

(ξ is codirected with the z axis.)

Although the support function of a hérisson is smooth, the hérisson itself (regarded as a surface) may have singular points. For singular points of B, we have $R_1 R_2 = 0$. If $R_1 R_2 \neq 0$, the hérisson B is a smooth surface in a neighbourhood of $\varphi(\xi)$ and the radii R_1 and R_2 coincide with the principal curvature radii (in the classical sense) of the surface B.

Suppose a convex body B with a smooth support function is a counterexample to the hypothesis, that is, we have everywhere $R_1 \leqslant C \leqslant R_2$. Consider now the hérisson $B^0 = B \otimes D^{-1}$, where D is the ball of radius C. Since the support function behaves additively with respect to Minkowski addition, the principal radii R_1^0 and R_2^0 of the hérisson B^0 satisfy the inequality $R_1^0 \leqslant 0 \leqslant R_2^0$. This means that the hérisson B^0 is a saddle surface.

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Conversely, let a hérisson B^0 be a saddle surface (except for its singular points). Owing to C^2-smoothness, its principal curvature radii R_1^0 and R_2^0 are bounded from below by a constant C. Then the convex body $B = B^0 \otimes D$ is a counterexample to the hypothesis. (As above, D is the ball of radius C.)

Thus the hypothesis is reduced to the existence problem of a hyperbolic hérisson (see Definition 4.1). A. V. Pogorelov tried to prove that such a surface does not exist.

The required hérisson (Figure 2.4) was constructed by Martinez-Maure [10]. It is a self-intersecting surface, which is obtained by gluing together graphs of two functions.

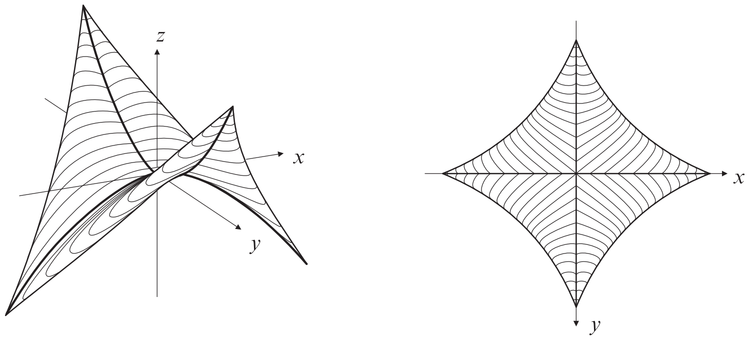

Figure 2.4

3 Virtual polytopes

The present section reviews some methods and constructions of [5], [16], [17], and [18] for dimension 3.

Denote by $\mathscr{P}$ the set of all compact convex polytopes in $\mathbb{R}^3$ with a fixed origin O (degenerate polytopes are also included). It is a semigroup with respect to the Minkowski addition $\otimes$. Denote by $\mathscr{P}^*$ the Grothendieck group of $\mathscr{P}$. The element of $\mathscr{P}^*$ which is inverse to $K \in \mathscr{P}$ is denoted by K^{-1}.

A function $F : \mathbb{R}^3 \to \mathbb{Z}$ is *polytopal* if it admits a representation of the form

$$F = \sum_i a_i I_{K_i},$$

where $a_i \in \mathbb{Z}$, $K_i \in \mathscr{P}$, and I_{K_i} is the indicator function of the polytope K_i:

$$I_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all polytopal functions is denoted by $\mathscr{M}$. It is endowed with two ring operations. The role of addition is played by the pointwise addition, denoted by $+$. The multiplication is generated by $\otimes$ and is denoted by the same symbol. The unit element of the ring $\mathscr{M}$ is obviously the function $E = I_{\{O\}}$.

Identifying convex compact polytopes with their indicator functions, we get an inclusion $\pi : \mathscr{P} \subset \mathscr{M}$. Keeping this identification in mind, we write K instead of I_K for convenience.

All elements of the semigroup $\pi(\mathscr{P})$ are invertible in $\mathscr{M}$. Hence the inclusion $\mathscr{P} \subset \mathscr{M}$ induces an inclusion $\mathscr{P}^* \subset \mathscr{M}$.

Definition 3.1. The image of the latter inclusion is called the group of *virtual polytopes*. For convenience, we denote it by the same letter $\mathscr{P}^*$.

Definition 3.2. Let K be a virtual polytope. Then there exist convex polytopes L and M such that $K = L \otimes M^{-1}$. The *support function* h_K of the virtual polytope K is defined to be the difference of support functions of L and M:

$$h_K = h_L - h_M.$$

This definition is consistent with other definitions of the support function (see [5] and [17]). Since we have defined the support function, we have the notion of *support oriented planes* of a virtual polytope as well.

Definition 3.3 ([16]). Let $K \in \mathscr{M}$, $K = \sum_i a_i K_i$ with $K_i \in \mathscr{P}$. Let $l_i(\xi)$ be the support plane to K_i with the outer normal vector ξ. The polytope $K_i^{\xi} = K_i \cap l_i(\xi)$ is called *the face with the normal vector ξ of the polytope K_i*, whereas the polytopal function $K^{\xi} = \sum_i a_i K_i^{\xi}$ is called *the face with the normal vector ξ of the polytopal function K*.

A face of a virtual polytope is a virtual polytope as well. 0-dimensional, 1-dimensional and 2-dimensional faces are called vertices, edges and facets respectively.

Definition 3.4. A *fan* Σ is a finite collection of compact spherical polygons on the unit sphere S^2 (possibly nonconvex and disconnected ones) such that

- $U, V \in \Sigma \Rightarrow U \cap V \in \Sigma$;
- $\bigcup \Sigma = S^2$;
- $U \neq V \in \Sigma \Rightarrow \operatorname{Relint} U \cap \operatorname{Relint} V = \varnothing$. (Relint stands for relative interior.)

The *fan* of a virtual polytope is defined below analogously to the classical definition of the outer normal fan. For a virtual polytope $K \in \mathscr{P}$, its *fan* Σ_K is the collection of spherically polytopal sets $\{\Sigma_K(v)\}$, where v ranges over the set of faces of K, and

$$\Sigma_K(v) = \operatorname{cl}\{\xi \mid K^{\xi} = v\}$$

(cl denotes the closure.)

These polytopal sets are called the *cells* of the fan. Similarly to the convex case, the support function of K is linear on each cell of Σ_K. And similarly to convex polytopes, the fan of a virtual polytope K can be defined as the minimal fan for which h_K is linear on each cell.

The 0-dimensional cells are called the *vertices* of the fan. The set of all vertices of a fan is denoted by Σ^0. It equals the set of normal vectors of all facets of K.

The collection of all 1-dimensional cells (= edges) of Σ is denoted by Σ^1 and is called the *skeleton* of Σ. The support of the skeleton *supp* Σ^1 is the union of all edges of Σ.

We say that K *fits* a fan Σ if Σ is a refinement of Σ_K. It means that h_K is linear on each cell of Σ.

For a unit vector ξ and a real number h, denote by $e(\xi, h)$ the plane whose equation is $(x, \xi) = h$. Note that a convex polytope $K \in \mathscr{P}$ is uniquely defined by the set $\Sigma^0 = \{\xi_i\}$ and the values $h_K(\xi_i) = h_i$. Indeed, the collection $\{e(\xi_i, h_i)\}$ is the collection of affine hulls of its facets.

However, this assertion fails when passing to virtual polytopes. In this case we have more freedom: we are free to choose a fan with vertices in $\{\xi_i\}$. The only thing we have to worry about is the consistency condition which is motivated by the following remark.

Remark 3.5. Let K be a virtual polytope. For any cell α of Σ_K with vertices $\{\xi_1, \dots, \xi_k\}$, the planes $e(\xi_1, h_1), \dots, e(\xi_k, h_k)$ have a common point. This point is the vertex of K corresponding to the cell α.

Consistency condition. We say that a fan Σ together with a function $h : \Sigma^0 \to \mathbb{R}$ satisfy the *consistency condition*, if for any cell α of Σ with vertices $\{\xi_1, \dots, \xi_k\}$, the planes $e(\xi_1, h_1), \dots, e(\xi_k, h_k)$ have a common point.

The following theorem demonstrates that virtual polytopes are not uniquely restored by the set $\{\xi_i\}$ and values h_i. It also shows that the cells in the virtual case may be non-convex, disconnected or of a complicated topological form.

Theorem 3.6 ([18]). *Let a fan Σ and a function $h : \Sigma^0 \to \mathbb{R}$ satisfy the consistency condition. Then there exists a unique virtual polytope K such that Σ_K fits Σ and $h_K(\xi_i) = h_i$.*

The following theorem yields a simple way of constructing examples of virtual polytopes.

Theorem 3.7. Construction of a virtual polytope related to an embedded simplicial complex. *Let $B \subset \mathbb{R}^3$ be a closed sphere-homeomorphic embedded* (*with possible self-intersections*) *simplicial complex generated by a set of triangles $\{T_i\}$. Suppose there exist a collection of normal vectors ξ_i of the triangles T_i and a fan Σ with vertices in $\{\xi_i\}$ such that the combinatorics of Σ is dual to that of B.* (*In particular, ξ_i and ξ_j are connected by an edge of Σ if and only if T_i and T_j share an edge in B.*)

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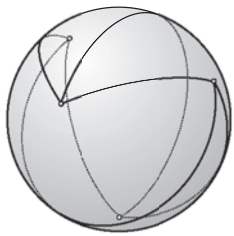
Then there exists a virtual polytope K such that

- *the set of closures of supports of facets of K coincides with the set of triangles* $\{T_i\}$, *and*
- $\Sigma_K = \Sigma$.

Here is an important example of this construction.

Example 3.8 ([17]). **Hyperbolic tetrahedron.** Let B be the complex of faces of a regular tetrahedron $\Delta \subset \mathbb{R}^3$. Choose the outward normal vectors of the triangles and connect their spherical images as is shown in Figure 3.9. The sphere is divided into four equal non-convex parts. Due to the above construction, we obtain a virtual polytope $\Delta' \neq \Delta$ which corresponds to this fan and this simplicial complex. It is a polytopal function which equals -1 everywhere on the tetrahedron except two fixed edges. On these edges the values of the function equal 0.



The fan of the hyperbolic tetrahedron

Figure 3.9

4 Hyperbolic virtual polytopes and hyperbolic hérissons

Let K be a virtual polytope or a smooth hérisson and let $h = h_K$ be its support function. For $\xi \in S^2$, let $e(\xi)$ be the plane defined by the equation $(x, \xi) = 1$. Consider the restriction of h to the plane $e(\xi)$ and denote by $F = F_K(\xi)$ the graph of the restriction. For a virtual polytope K, the surface F is piecewise linear. Its vertices and edges correspond to those of Σ_K intersected with the open hemisphere with the pole ξ. For a smooth hérisson K, the surface F is smooth.

Definition 4.1. A virtual polytope K (or a smooth hérisson) is called *hyperbolic* if $F_K(\xi)$ is a saddle surface for any $\xi \in S^2$.

Recall that a piecewise linear (or any other non-smooth surface) F is called *a saddle surface* if there is no plane cutting a bounded connected component off F (see [4]).

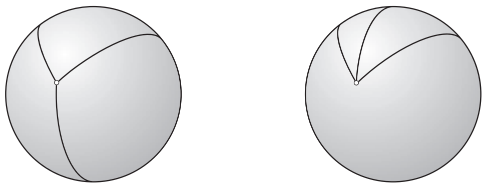
Let $\Xi = \{\xi_i\}$ be a collection of points on S^2 such that each open hemisphere contains at least one point from the collection.

Proposition 4.2. *A hérisson K is hyperbolic if and only if $F_K(\xi)$ is a saddle surface for any $\xi \in \Xi$. In this case, $F_K(\xi)$ is saddle for any other $\xi \in S^2$ as well.*

Proof. Indeed, if the function $h|_{e(\xi)}$ is saddle at the point $X \in e(\xi)$, then for any other ξ', the function $h|_{e(\xi')}$ is saddle at the point $X' \in e(\xi')$ provided by $\frac{X}{|X|} = \frac{X'}{|X'|}$. □

Remark 4.3. Martinez-Maure has already given two definitions of weak and strong hyperbolicity for virtual polytopes (see [11]). The above definition is equivalent to that of weak hyperbolicity.

Definition 4.4. We say that a vertex ξ of a fan Σ is *nonconvex* if for a neighbourhood $U(\xi)$ of ξ, $\Sigma^1 \cap U(\xi)$ lies on one side of a great circle containing ξ.



a convex vertex a non-convex vertex

Figure 4.5

Proposition 4.6. *Let K be a virtual polytope. Suppose that each vertex of its fan Σ_K has exactly* 3 *adjacent edges. Then the two assertions are equivalent*:

- *K is hyperbolic.*
- *Each vertex of Σ_K is non-convex.*

Proof. A vertex with exactly 3 adjacent edges can be cut off the surface $F_K(\xi)$ if and only if its image in the fan is convex. It remains to observe that if F is non-hyperbolic, there exists a plane cutting a bounded component off F containing exactly one vertex. □

Corollary 4.7. *The hyperbolic tetrahedron* (*Example* 3.8) *is a hyperbolic virtual polytope.*

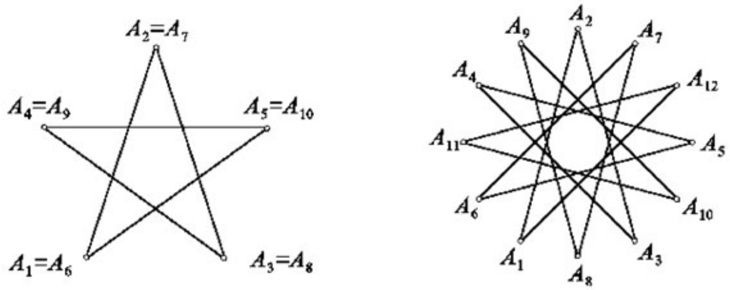
5 Example of a hyperbolic virtual polytope with N cross-caps

The desired virtual polytope $\mathscr{K}$ is constructed as follows. First, we construct an embedded simplicial complex. It consists of its upper and lower central parts (which look like two folded stars) and of N discrete cross-caps (Figure 5.9). The triangles are oriented, and the endpoints of their normal vectors serve as vertices of a fan (Figure 5.8). By Theorem 3.7, there exists a virtual polytope related to the complex.

Planar stars. Let N be even, $N \geqslant 4$.

Case 1. *N*/2 is odd. Consider a (planar) proper $N/2$-gon P. By the *star with N rays* (based on P) we mean the broken line consisting of all longest diagonals of P. Each of the diagonals is taken twice, so the broken line has a double self-overlapping. The vertices $\{A_i\}$ of the star (i.e., the vertices of P taken twice) are enumerated clockwise.

Case 2. *N*/2 is even. Consider a (planar) proper N-gon P. By the *star with N rays* (based on P) we mean the broken line consisting of all diagonals of P connecting a vertex of P with a vertex lying next to the opposite one. Unlike the previous case, the broken line is not self-overlapping. The vertices $\{A_i\}$ of the star (i.e., the vertices of P) are enumerated clockwise.



A star with odd *N*/2 A star with even *N*/2

Figure 5.1

Central collections of triangles. Fix a Cartesian coordinate system (O, x, y, z). All the vectors in $\mathbb{R}^3$ forming an angle with the z axis less than $\pi/2$ will be referred as "*vectors looking upwards*". If the angle is more than $\pi/2$, we say that the vector *looks downwards*. If the angle equals 0, the vector is called *horizontal*.

Consider a star with N rays ($N = 4, 6, 8, \dots$) lying in the plane $z = 0$ and centered at O. Denote its vertices by $\{A_i'\}_{i=1}^N$. Let $A_i' = (x_i, y_i, 0)$.

Assuming that ε is small (its precise value is discussed later). Introduce a new collection of points: For odd $N/2$, put $A_i = (x_i, y_i, (-1)^i\varepsilon)$, $i = 1, \dots, N$. For even $N/2$, put $A_i = (x_i, y_i, (-1)^i\varepsilon)$, $\bar{A}_i = (x_i, y_i, (-1)^{i+1}\varepsilon)$, $i = 1, \dots, N$.

Remark 5.2. Here and in the sequel, given an ordered set of any objects $\{X_i\}_{i=1}^M$ we assume that for any $k \in \mathbb{Z}$, we have $X_k = X_i$ if $k \equiv i \pmod{M}$.

Consider now two collections of oriented triangles which will form the central part of the desired virtual polytope.

Case 1. *N*/2 is odd. Put

$$J^{+}(\varepsilon) = \{T_i^{+}\}_{i=1}^{N} = \{OA_iA_{i+1}\}_{i=1}^{N}$$

and

$$J^{-}(\varepsilon) = \{T_i^{-}\}_{i=1}^{N} = \{A_iOA_{i+1}\}_{i=1}^{N}.$$

These collections differ only by orientation: the normal vectors of J^{+} look upwards whereas the normal vectors of J^{-} look downwards.

Remark 5.3. We indicate an orientation of a triangle in two ways: by indicating its normal vector ξ and by the order of its vertices.

Case 2. *N*/2 is even. Put

$$J^{+}(\varepsilon) = \{T_i^{+}\}_{i=1}^{N} = \{OA_iA_{i+1}\}_{i=1}^{N}$$

and

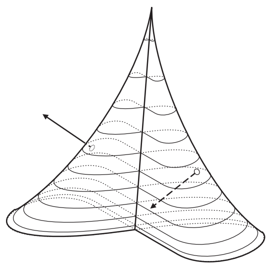
$$J^{-}(\varepsilon) = \{T_i^{-}\}_{i=1}^{N} = \{\bar{A}_iO\bar{A}_{i+1}\}_{i=1}^{N}.$$

As in the previous case, the normal vectors of the first (respectively, second) collection look upwards (respectively, downwards).

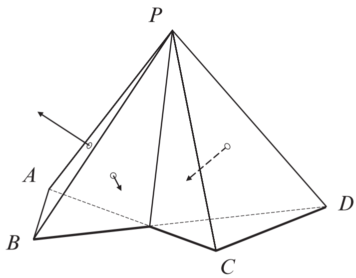
Discrete cross-cap. A general construction. A *smooth cross-cap* is a smooth oriented self-intersecting saddle surface with one singular point which looks like the one in Figure 5.4.

Smooth cross-caps with *infinite horn* (in terminology of [4], when the singular point is on infinity) were investigated by A. Verner [25]. Cross-caps with finite horns were used by Y. Martinez-Maure [10] as a crucial tool for constructing his counterexample.

A *discrete cross-cap* consists of four oriented triangles $\{APB, DBP, DPC, ACP\}$ (Figure 5.4). We say that the cross-cap is *based* on the rectangle $ABCD$.



A smooth cross-cap



A discrete cross-cap

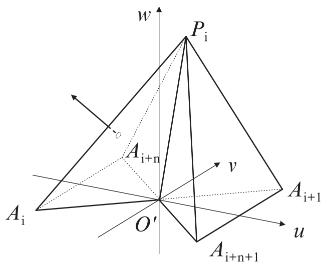
Figure 5.4

Construction of N cross-caps glued to the central parts. Here we construct N cross-caps based on the rectangles $A_iA_{i+N/2}A_{i+1}A_{i+N/2+1}$ for odd $N/2$ and on the rectangles $A_i\bar{A}_iA_{i+1}\bar{A}_{i+1}$ for even $N/2$.

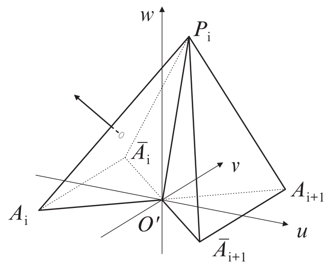
Case 1. $N/2$ is odd. Denote $n = N/2$. Choose another Cartesian coordinate system (O', u, v, w) related to the rectangle $A_iA_{i+n}A_{i+1}A_{i+n+1}$. That is, $O' = [A_iA_{i+1}] \cap [A_{i+n}A_{i+n+1}]$. The plane $w = 0$ coincides with the $\mathrm{aff}(A_iA_{i+n}A_{i+1}A_{i+n+1})$. The direction of the w axis is chosen to make $w(O) \leqslant 0$. The v axis is codirected with the vector $\overrightarrow{A_iA}_{i+n}$. The u axis is codirected with the vector $(-1)^i\,\overrightarrow{A_iA}_{i+n+1}$.

Let $P_i = P_i(l, \delta) = (o, \delta, l)$. ($l$ is assumed to be big, but the precise values of l and δ will be specified later.) The required cross-cap $C_i = C_i(\delta, l)$ is based on the rectangle $A_iA_{i+1}A_{i+n}A_{i+n+1}$. Namely,

$$C_i = \{A_iA_{i+n}P_i, A_{i+1}A_iP_i, A_{i+n+1}P_iA_{i+n}, A_{i+n+1}A_{i+1}P_i\} = \{S_i, R_i^+, R_i^-, S_i'\}.$$



N/2 is odd



N/2 is even

Figure 5.5

Case 2. $N/2$ is even. Choose a Cartesian coordinate system $(O'u, v, w)$ related to the rectangle $A_iA_{i+1}\bar{A}_i\bar{A}_{i+1}$. That is, $O' = [A_iA_{i+1}] \cap [\bar{A}_i\bar{A}_{i+1}]$. The plane $w = 0$ coincides with the $\mathrm{aff}(A_iA_{i+1}\bar{A}_i\bar{A}_{i+1})$. The direction of the w axis is chosen to make $w(O) \leqslant 0$. The v axis is codirected with the vector $\overrightarrow{A_i\bar{A}_i}$. The u axis is codirected with the vector $(-1)^i\,\overrightarrow{A_i\bar{A}}_{i+1}$.

Let $P_i = P_i(l, \delta) = (o, \delta, l)$. (Again, l is assumed to be big.) Consider the cross-cap $C_i = C_i(\delta, l)$, based on the rectangle $A_iA_{i+1}\bar{A}_i\bar{A}_{i+1}$. Namely,

$$C_i = \{A_i\bar{A}_iP_i, A_{i+1}A_iP_i, \bar{A}_{i+1}P_i\bar{A}_i, \bar{A}_{i+1}A_{i+1}P_i\} = \{S_i, R_i^+, R_i^-, S_i'\}.$$

Figure 5.6 presents two sequential cross-caps. The vectors $\overrightarrow{O'P_i}$ look in turn upwards and downwards.

Note that the normal vectors of R_i^+ look upwards (with respect to the old coordinate system (x, y, z)), whereas the normal vectors of R_i^- look downwards. The normal vectors of triangles S_i and S_i' are horizontal. The triangles S_i and S_{i+1}' have a mutual edge.

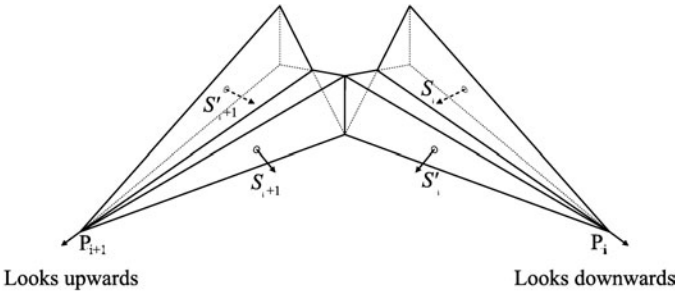


Figure 5.6

Now consider the union of all the above constructed collections of triangles

$$\{T_i^+\}_{i=1}^N \cup \{T_i^-\}_{i=1}^N \cup \bigcup_{i=1}^N C_i.$$

After specifying below the pairs of adjacent triangles, we obtain an embedded simplicial complex which is denoted by $\mathcal{C}$.

T_i^+ is adjacent to T_{i+1}^+, T_{i-1}^+ and R_i^+; T_i^- is adjacent to T_{i+1}^-, T_{i-1}^- and R_i^-;
R_i^+ is adjacent to S_i, S_i' and T_i^+; R_i^- is adjacent to S_i, S_i' and T_i^-;
S_i is adjacent to R_i^+, S_{i-1}' and R_i^-; S_i' is adjacent to R_i^+, S_{i+1} and R_i^-.

Theorem 5.7. *For an appropriate choice of* ε, δ, *and* l, *the complex* $\mathcal{C}$ *generates* (*by Theorem* 3.7) *a hyperbolic virtual polytope* $\mathcal{H}$ *with* N *discrete cross-caps. For* $N = 4$, *the polytope coincides with the example constructed in* [11].

Proof. We have already specified the orientations of the triangles. Mark on S^2 their spherical images (which are denoted by the same letters as the triangles) and connect them by geodesic segments as shown in Figure 5.8, the right part. By Theorem 3.7, the fan obtained, together with the complex $\mathcal{C}$, yields the desired virtual polytope. The opposite side of the fan is equal to the front one. Two things should be noted:

1. For a proper choice of ε, δ, and l, the geodesic lines connecting the vertices of the fan do not intersect (except for the vertices). This is demonstrated through the pair of fans (Figure 5.8). The first fan reflects the limit situation as $\varepsilon \to 0$ and $l \to \infty$ (all the points T_i^+ coincide, and $S_i = S_i'$. Besides, the angles between T_iR_i and R_iS_i are equal to $\pi/2$.) Its edges obviously do not intersect. To obtain a fan of the required type, it suffices to move somewhat the vertices of the first fan. The values of ε, δ and l can be easily restored from the fan.

2. The virtual polytope obtained is hyperbolic by Proposition 4.6, since each of its vertices is nonconvex. □

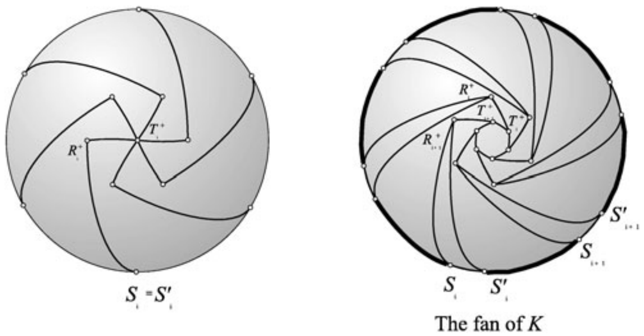


Figure 5.8

From above, the polytope $\mathscr{K}$ looks like in Figure 5.9.

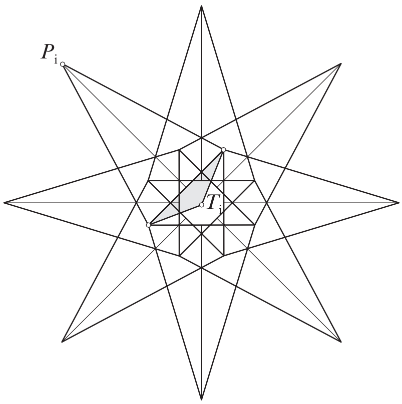


Figure 5.9

6 Hyperbolic smoothing

In the present section, we construct a C^∞-smooth hyperbolic approximation $\tilde{h}$ of the support function $h = h_{\mathscr{K}}$ of the above virtual polytope $\mathscr{K}$. This is done as follows.

- Step 1 demonstrates that the polyhedral surface $F_K(\xi)$ admits a local approximation of the desired type.
- Step 2 gives a global approximation of the surfaces $F_K(0, 0, \pm 1)$.
- Step 3 gives an approximation of $F_K(\xi)$ for a horizontal vector ξ.
- Finally, Step 4 makes all above approximations mutually consistent, i.e., generated by one and the same smooth function $\tilde{h}$ defined on the sphere.

Remark. In approximating the above polyhedral surfaces, we do not change them at the points lying far from the edges. Along the edges (but far from the vertices) we replace the surface either by a cylinder or by a cone.

In particular, this means that contrary to the example given by Y. Martinez-Maure, the hérisson constructed below has singularities not only at the endpoints of its horns. Indeed, for each planar, conical, or cylindrical part of the approximating surface, we have $R_1 R_2 = 0$.

A *cylinder* is a set of points that is invariant under all translations parallel to a line l. A *cone* with a vertex A is a set of points that is invariant under homotheties with center in A.

Now we prove the central theorem of the paper.

Theorem 6.1. *For any even $N = 4, 6, 8, \dots$, there exists a C^∞-smooth hyperbolic hérisson containing N cross-caps.*

By the trick of Martinez-Maure and Pogorelov (*Section* 1), *each such hérisson generates a counterexample to the hypothesis.*

Proof. Step 1. Let $F = F_K(\xi)$ for $\xi = (0, 0, \pm 1)$. Consider a neighbourhood U of a vertex A of the surface F. Denote the adjacent edges by L_1, L_2, and L_3, as is shown in Figure 6.2. Let e be the plane containing L_2 and orthogonal to $\operatorname{aff}(L_1 \cup L_3)$. Choose two planar C^∞-smooth convex curves $f = f(A)$ and $g = g(A)$ such that f is inscribed in $e \cap F$ and coincides with $e \cap F$ outside a small neighbourhood of A, and g is inscribed in $(L_1 \cup L_2)$ and coincides with $(L_1 \cup L_2)$ outside a small neighbourhood of A (Figure 6.2).

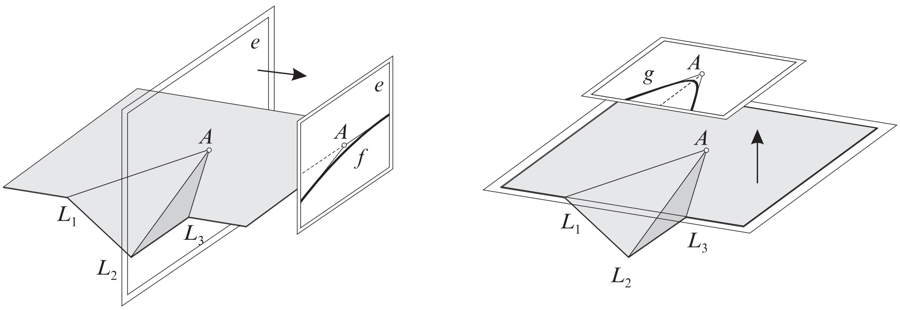


Figure 6.2

In the neighbourhood U, we have $F = \{u(e \cap F) \mid u$ is a translation such that $u(A) \in L_1 \cup L_2\}$. Now replace F (in U) by the smooth surface $\bar{F} = \{u(f) \mid u$ is a translation such that $u(A) \in g\}$.

The edges L_1, L_2, and L_3 are replaced by the cylinders c_1, c_2, and c_3. The cylinders c_1 and c_3 are determined by the choice of f. The cylinder c_2 is determined by the choice of g.

Step 2. In constructing a global approximation of $F = F_K(0, 0, \pm 1)$, we must take into account the dependence between g and f for different vertices: for adjacent vertices $A_{1,2}$, we must construct the same cylinders approximating the edge $A_1 A_2$.

The following simple scheme shows that there exists a global choice of g and f for all vertices, which yields a global approximation of F and demonstrates freedom of choice.

1. We may choose independently g and f for all R_i^+.

2. Now, $f(T_i)$ is determined by $f(R_i^+)$ and $g(T_i)$ is determined by $f(R_{i+1}^+)$.

This gives us an approximation of h everywhere on the sphere except for a neighbourhood of the horizontal circle. Moreover, we are free to choose independently any (sufficiently narrow) cylinders, which replace $S_i R_i^\pm$ and $S_i' R_i^\pm$.

Step 3. Consider a horizontal vector ξ such that the hemisphere with the pole ξ contains the edge $S_i' S_{i+1}$ for certain i. Consider the surface $F = F_K(\xi)$ in a neighbourhood U of the point A that corresponds to S_i'. Denote the adjacent edges by L_1, L_2, and L_3. We have already a smooth approximation of F along the edges L_1 and L_2, which comes from Steps 1 and 2, but unlike the previous case, the edges are approximated by cones with vertices at the point A.

As in Step 1, we consider the plane e containing L_2 and orthogonal to $\operatorname{aff}(L_1 \cup L_3)$. Unlike Step 1, the sections of F parallel to e will be chosen not equal but homothetic. We replace F (in the neighbourhood U) by the smooth surface $\bar{F} = \{\Lambda_k(u_X(f)) \mid u$ is a translation such that $u(A) \in g\}$.

Λ_k is a homothety with the origin $u(A)$ and the coefficient $k = k(u)$.

g is a smooth curve inscribed in $L_1 \cup L_2$. Here is the precise construction.

Assume for simplicity that in a coordinate system (u, v, w) (Figure 6.3), we have

$$\operatorname{aff}(L_1, L_3) \text{ is the plane } w = 0;\ A = (0,0,0);$$

$$L_1 \cup L_3 = \{(u, v, 0) \mid |u| = v\}; \quad L_2 = \{(0, v, w) \mid 0 \leqslant v = -w\}.$$

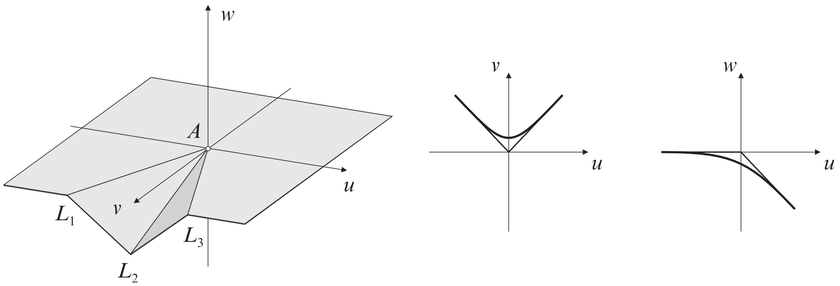


Figure 6.3

Choose two C^∞-smooth functions f and k such that the convex graphs of k and f are inscribed in the graphs of the functions $v = |u|$ and $v = -\frac{u+|u|}{2}$ respectively. Note that for any constant C, the function f can be chosen so that the condition $f'' \neq 0$ implies $f \geqslant Cf'$. Put

$$\mathscr{F}(u, v) = k(u) f\left(\frac{v}{k(u)} - 1\right).$$

The graph of the function $\mathscr{F}$ is the desired smooth hyperbolic local approximation. The edges L_1 and L_3 are approximated by cones (determined by the choice of f), whereas L_2 is approximated by a cylinder (determined by the choice of k).

Ordinary calculations that check hyperbolicity show that

$$\det\begin{pmatrix} \mathscr{F}''_{uu}(u,v) & \mathscr{F}''_{uv}(u,v) \\ \mathscr{F}''_{vu}(u,v) & \mathscr{F}''_{vv}(u,v) \end{pmatrix} = \frac{k''(u)}{k^2(u)} f''\left(\frac{v}{k(u)}-1\right)\left[k(u)f\left(\frac{v}{k(u)}-1\right) - f'\left(\frac{v}{k(u)}-1\right)v\right] \leqslant 0.$$

Indeed, if $f''\left(\frac{v}{k(u)}-1\right)=0$, the inequality is obviously valid. If $f''\left(\frac{v}{k(u)}-1\right)\neq 0$ and $k''\neq 0$, then k and v are bounded above and below by some positive numbers. Therefore, for a proper choice of f, we have

$$k(u)f\left(\frac{v}{k(u)}-1\right) - f'\left(\frac{v}{k(u)}-1\right)v \geqslant 0,$$

which, together with $k''(u)f''\left(\frac{v}{k(u)}-1\right)\leqslant 0$, implies the required inequality.

Analogously to the previous step, the function k determines a cylinder that approximates the edge L_2. Construct now an approximation of the surface in a neighbourhood of S_{i+1} with the same cylinder for L_2.

Step 4. Consider a narrow belt about the horizontal great circle $S^1 \subset S^2$. By Step 3, construct a smooth hyperbolic approximation of $h_{\mathscr{K}}$ in the belt such that the edges $R_i^{\pm}S_i$ and $R_i^{\pm}S_i'$ are approximated by cones. Passing to $F = F_{\mathscr{K}}((0,0,\pm 1))$, we obtain cylindrical approximations of the edges $R_i^{\pm}S_i$ and $R_i^{\pm}S_i'$. By Step 2, this can be extended to a global approximation of F, which yields a global smooth hyperbolic approximation of $h_{\mathscr{K}}$. □

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