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# On hyperbolic virtual polytopes and hyperbolic fans

Gaiane Panina*

*Institute for Informatics and Automation,*
*St.Petersburg, 199178, Russia*

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**Abstract:** Hyperbolic virtual polytopes arose originally as polytopal versions of counterexamples to the following A.D.Alexandrov's uniqueness conjecture:
*Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant $C$, at every point of $\partial K$, we have $R_1 \leq C \leq R_2$ then $K$ is a ball. ($R_1$ and $R_2$ stand for the principal curvature radii of $\partial K$.)*
This paper gives a new (in comparison with the previous construction by Y.Martinez-Maure and by G.Panina) series of counterexamples to the conjecture. In particular, a hyperbolic virtual polytope (and therefore, a hyperbolic hérisson) with odd an number of horns is constructed. Moreover, various properties of hyperbolic virtual polytopes and their fans are discussed.


*Keywords: Virtual polytope, saddle surface, hérisson*
*MSC (2000): 52A15, 52B70, 52B10*

## 1 Introduction

In this paper, we study *hyperbolic virtual polytopes.* Figuratively speaking, hyperbolic virtual polytopes relate to the convex ones in the same way as convex surfaces relate to saddle ones. As is known, there exists no closed saddle polytopal surface. Still, non-trivial hyperbolic virtual polytopes do exist and this is probably the most remarkable fact known about them. Non-trivial hyperbolic virtual polytopes appeared originally as an auxillary construction for various counterexamples to the following A.D.Alexandrov's uniqueness hypothesis:

*Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant $C$, in each point of $\partial K$, we have $R_1 \leq C \leq R_2$, then $K$ is a ball. ($R_1$ and $R_2$ stand for the principal curvature radii*

* E-mail: panina@iias.spb.su

*of $\partial K$).*

For a long time mathematicians were certain about the correctness of the hypothesis, but obtained only some partial results. Recently, Y.Martinez-Maure [5] has given a counterexample. First, he demonstrated that each smooth *hyperbolic hérisson* (see Section 2) generates a desired counterexample. Next, he presented such an example (see Fig. 1). It is a smooth hyperbolic surface with four *horns* (i.e., points where the surface is neither hyperbolic no smooth), given by an explicit formula.

Surprisingly, this counterexample proved to be not unique: a series of counterexamples was given by the author of the paper (see [9]). Using a different technique, she constructed smooth hyperbolic hérissons with any even number of horns greater than 4. The present paper continues this study and demonstrates that they are even more various.

The paper is organized as follows. Sections 2 and 3 give necessary definitions and examples of *virtual polytopes* (which are, roughly speaking, Minkowski differences of convex polytopes). In addition, Section 2 recalls briefly the notion of *hérissons* (i.e., Minkowski differences of smooth convex bodies). The definition of *hyperbolic virtual polytopes* (i.e., virtual polytopes such that the graph of the support function is a saddle surface) are presented in Section 4. Convex poytopes and hyperbolic polytopes are compared in Theorem 4.4.

Section 5 studies the fans of simplicial hyperbolic virtual polytopes. The edges of such a fan admit a *proper coloring*, which encodes important properties of the virtual polytope. For instance, a cell of the fan corresponds to a horn of the polytope, if and only if the color changes twice as while going around the cell (Theorem 5.3).

Theorem 5.7 demonstrates a way of adding a new horn to a hyperbolic polytope. This is called a $\mathcal{C}$-operation (or a $\mathcal{S}$-operation) and arises from some special refinement of the fan of the original polytope. This trick is used in Section 6, which gives advanced examples of hyperbolic virtual polytopes, in particular, with an odd number of horns.

Note that it is impossible to add a horn to any hyperbolic polytope known before ([6] and [9]), so we have to construct new ones.

Hyperbolic virtual polytopes can be classified in a reasonable way by the number of horns. However, there exists a finer classification since each hyperbolic polytope generates in a natural way an arrangement of oriented great semicircles (each horn gives a semicircle). We are bound by the case when the semicircles (and therefore, the horns) admit a natural ordering. This allows us to assign to a polytope $K$ a *necklace* (i.e., a circular sequence) consisting of $N$ signs " $+$ " and " $-$ " ($N$ stands for the number of horns).

By Theorem 6.1, each necklace with more than three changes of sign is realizable as a hyperbolic polytope.

Much room is left here for further study: applying $\mathcal{C}$- and $\mathcal{S}$-operations one can obtain further types of hyperbolic polytopes, in particular, with new combinatorial type of semicircle arrangement. However, we leave this beyond the paper.

In Section 7, we show that each polytope constructed in Section 6 admits a hyperbolic smoothing and therefore yields a counterexample to the A.D. Alexandrov's hypothesis.

As in [9], we smooth not the surface of the virtual polytope, but the collection of graphs of its support function.

## 2 Virtual polytopes and hérissons: basic notations.

Virtual polytopes (introduced by A.Pukhlikov, A.Khovanskii [4], appeared also in a natural way in P.McMullen's polytope algebras [7]) and can be represented in four different ways.

- Virtual polytopes are elements of the Grothendieck group of the semigroup of convex polytopes $\mathcal{P}$ in $\mathbb{R}^n$ equipped with the Minkowski addition $\otimes$. I.e., they are formal expressions of type $K \otimes L^{-1}$, where $K, L \in \mathcal{P}$.
- Virtual polytopes are *polytopal functions* (Definition 2.2), i.e., finite linear combinations of characteristic functions of convex polytopes. So it makes sense to speak of the *value* of a polytope $K$ at a point $X \in \mathbb{R}^n$.
- Virtual polytopes are defined by their *support functions*, i.e., piecewise linear positively homogeneous functions defined on $\mathbb{R}^n$ (Definition 2.3).
- A virtual polytope is a pair of type (a closed polytopal surface in $\mathbb{R}^n$ with cooriented facets; a spherical fan) (see Theorem 2.10).

We now give a detailed explanation of the items (restricting ourselves to dimension $n = 3$).

Denote by $\mathcal{P}$ the set of all compact convex polytopes in $\mathbb{R}^n$ (degenerate polytopes are also included). It is a semigroup with respect to the Minkowski addition $\otimes$.

Denote by $\mathcal{P}^*$ the Grothendieck group of $\mathcal{P}$. The element of $\mathcal{P}^*$ that is inverse to $K$ is denoted by $K^{-1}$.

A function $F : \mathbb{R}^3 \to \mathbb{Z}$ is *polytopal* if it admits a representation of the form

$$F = \sum_i a_i I_{K_i},$$

where $a_i \in \mathbb{Z}$, $K_i \in \mathcal{P}$, and $I_{K_i}$ is the indicator function of the polytope $K_i$:

$$I_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all polytopal functions $\mathcal{M}$ is endowed with two ring operations. The role of addition is played by the pointwise addition, denoted by $+$. The multiplication is generated by $\otimes$ and is denoted by the same symbol.

The unit element of the ring $\mathcal{M}$ is obviously the function $E = I_{\{O\}}$.

Identifying convex compact polytopes with their indicator functions, we get an inclusion $\pi : \mathcal{P} \subset \mathcal{M}$. Keeping this identification in mind, we write for convenience $K$ instead of $I_K$.

Due to the following theorem, all elements of the semigroup $\pi(\mathcal{P})$ are invertible in $\mathcal{M}$.

**Theorem 2.1.** *(On Minkowski inversion)* [4]

*For any convex polytope $K$, we have*

$$(-1)^{\dim K} I_{Relint(sK)} \otimes K = E,$$

*where $s$ is the central symmetry mapping (with respect to the origin $O$) , $Relint(sK)$ is the relative interior of the polytope $sK$ (i.e., the interior taken in the affine hull of $K$).*

Hence the inclusion $\mathcal{P} \subset \mathcal{M}$ induces an inclusion $\mathcal{P}^* \subset \mathcal{M}$.

**Definition 2.2.** The image of the latter inclusion is called the group of *virtual polytopes*. For convenience we denote it by the same letter $\mathcal{P}^*$.

**Definition 2.3.** Let $K$ be a virtual polytope. Then there exists convex polytopes $L$ and $M$ such that $K = L \otimes M^{-1}$. The *support function* $h_K$ of the virtual polytope $K$ is defined to be the pointwise difference of support functions of $L$ and $M$:

$$h_K = h_L - h_M.$$

**Remark 2.4.** Recall that the support function of a convex polytope is piecewise linear with respect to a fan. By a *fan* we mean a splitting of $\mathbb{R}^n$ in a union of polytopal cones with a common apex at $O$. In the sequel, we sometimes speak of (and draw) the intersection of the fan with the unite sphere $S^{n-1}$ centered at $O$. Thus, the cones correspond to spherical polytopes (spherical cells).

**Definition 2.5.** [8] Let $K = \sum_i a_i K_i$ with $K_i \in \mathcal{P}$. Let $e_i(\xi)$ be the support hyperplane to $K_i$ with the outer normal vector $\xi$. The polytope $K_i^\xi = K_i \cap e_i(\xi)$ is called *the face of the polytope $K_i$ with the normal vector $\xi$*, whereas the polytopal function $K^\xi = \sum_i a_i K_i^\xi$ is called *the face of the polytopal function $K$ with the normal vector $\xi$.*

A face of a virtual polytope is a virtual polytope as well. The 0-dimensional, 1-dimensional and 2-dimensional faces are called vertices, edges and facets respectively. (By the *dimension* of a virtual polytope we mean the dimension of the affine hull of its support.)

Similarly to faces of convex polytopes, virtual faces behave linearly with respect to the Minkowski addition:

**Theorem 2.6.** [8] *In the above notation,*

$$K_1^\xi \otimes K_2^\xi = (K_1 \otimes K_2)^\xi.$$

**Definition 2.7.** A point $X$ is called a *boundary point* of a polytope $K$, if $x \in cl(supp(K^\xi))$ for some $\xi \in S^2$ such that $\xi$ is not orthogonal to $aff(K)$. ($cl$ denotes the closure.)

**Definition 2.8.** A *fan* $\Sigma$ is a finite collection of compact spherical polytopes on the unit sphere $S^2$ (possibly nonconvex ones) such that

- $U, V \in \Sigma \Rightarrow U \cap V \in \Sigma$;

- $\bigcup_{\Sigma} U = S^2$;
- $U \neq V \in \Sigma \Rightarrow RelintU \cap RelintV = \emptyset$.

The fan of a virtual polytope is defined below analogously to the classical definition of the outer normal fan.

**Definition 2.9.** For a virtual polytope $K \in \mathcal{P}^*$, its *fan* $\Sigma_K$ is the collection of spherically polytopal sets $\{\Sigma_K(\nu)\}$, where $\nu$ ranges over the set of faces of $K$, and

$$\Sigma_K(\nu) = cl(\{\xi | K^{\xi} = \nu\})$$

($cl$ denotes the closure.)

These polytopal sets are called the *cells* of the fan. Similarly to the convex case, the support function of $K$ is linear on each cell of $\Sigma_K$. And similarly to convex polytopes, the fan of a virtual polytope $K$ can be defined as the minimal fan for which $h_K$ is linear on each cell. In addition, we have the usual duality: $k$-dimensional cells of $\Sigma_K$ correspond to $(3-k-1)$-dimensional faces of $K$.

The 0-dimensional cells are called the *vertices* of the fan.

From now on, we assume that $n = 3$, and deal with 3-dimensional virtual polytopes.

A virtual polytope is said to be *simplicial* if each of its facets is a virtual triangle (see Section 3). Each simplicial virtual polytope $K$ yields in a natural way a sphere-homeomorphic simplicial complex $\mathbf{C}_K$ which is generated by the collection of triangles $\{cl(supp(K^{\xi}))|\ \xi \in S^2;\ dim(K^{\xi}) = 2\}$.

Alternatively, given a simplicial complex $\mathbf{C}$, it is sometimes possible to associate with $\mathbf{C}$ a virtual polytope. Moreover, sometimes it is possible to associate many different virtual polytopes (see Figure 5). The general construction is given in the following theorem.

**Theorem 2.10. Construction of a virtual polytope related to an immersed simplicial complex [8, 9].**

*Let $\mathbf{C}$ be a closed sphere-homeomorphic immersed (with possible self-intersections) in $\mathbb{R}^3$ simplicial complex generated by a set of triangles $\{T_i\}$.*

*Suppose there exists a collection of normal vectors $\xi_i$ of the triangles $T_i$ and a spherical fan $\Sigma$ with vertices in $\{\xi_i\}$ satisfying the two conditions:*

***main condition*** *The combinatorics of $\Sigma$ is dual to that of $\mathbf{C}$ . (In particular, $\xi_i$ and $\xi_j$ are connected by an edge of $\Sigma$ if and only if $T_i$ and $T_j$ share an edge in $\mathbf{C}$.)*

***condition for complexes with parallel adjacent facets*** *If two adjacent facets $T_i$ and $T_j$ of the complex are parallel (and therefore, have opposite normal vectors $\xi_i$ and $\xi_j$), then the points $\xi_i$ and $\xi_j$ are connected by an edge (respectively, edges) of $\Sigma_K$, which is orthogonal to the mutual edge (respectively, mutual edges) of the facets.*

*Then there exists a virtual polytope $K$ such that*

- *the set of $\{cl(supp(K^{\xi}))|\ \xi \in S^2;\ dim(K^{\xi}) = 2\}$ coincides with the set $\{T_i\}$, and*
- *$\Sigma_K = \Sigma$.*

**Remark 2.11.** Given a virtual polytope $K$ in $\mathbb{R}^3 = (x, y, z)$, the vertices of the associated complex $\mathbf{C}_K$ can be restored by the support function $h = h_K$ as follows. Let a vertex $A$ of $\mathcal{C}_K$ correspond by duality to a cell $\alpha$ of the fan $\Sigma_K$. Then $A = ((h|_\alpha)'_x, (h|_\alpha)'_y, (h|_\alpha)'_z)$, where $h|_\alpha$ is the restriction of $h$ to the cell $\alpha$.

Similar geometric realization of Minkowski differences of smooth convex bodies makes sense as well. It can be traced in the early paper [1] by A.D. Alexandrov. Let $h : S^2 \to \mathbb{R}$ be a smooth function. By the *hérisson* $H$ with the support function $h$ (see [11]), we mean the envelope of the family of planes $\{e_H(\xi)\}_{\xi \in S^2}$, where the plane $e_H(\xi)$ is defined by the equation

$$(x, \xi) = h(\xi).$$

It is a sphere-homeomorphic surface with possible self-intersections and self-overlapings. We say that a hérisson $H$ is *smooth* if its support function is smooth.

As a set of points, a hérisson $H$ coincides with the image of the mapping

$$\phi : S^2 \longrightarrow \mathbb{R}^3,$$

$$(x, y, z) \longrightarrow (h'_x(x, y, z), h'_y(x, y, z), h'_z(x, y, z)).$$

Analogously to the classical convex case, *the principal curvature radii* $R_1$ and $R_2$ of a hérisson $H$ at a point $\xi \in S^2$ (or at the point $\phi(\xi) \in H$) are the eigenvalues of the matrix

$$\begin{pmatrix} h''_{xx}(\xi) & h''_{xy}(\xi) \\ h''_{yx}(\xi) & h''_{yy}(\xi) \end{pmatrix}.$$

($\xi$ is codirected with the $z$ axis.)

Although the support function of a hérisson is smooth, the hérisson itself (regarded as a surface) may have singular points. They appear whenever $R_1 R_2 = 0$.

If $R_1 R_2 \neq 0$, the hérisson $B$ is a smooth surface in a neighbourhood of $\phi(\xi)$ and the radii $R_1$ and $R_2$ coincide with the principal curvature radii of the surface $H$.

Martinez-Maure observed that a body $K$ together with a constant $C$ give a counterexample to A.D. Alexandrov's hypothesis if and only if the hérisson $K \otimes B_C^{-1}$ is *hyperbolic*, i.e., $R_1 R_2 \leq 0$ everywhere ($B_C$ stands for a ball of radiuce $C$). An example of a hyperbolic hérisson presented by Martinez-Maure [5] is a surface (see Fig. 1) obtained by gluing together graphs of two explicitly given functions.

## 3 Examples of virtual polytopes

**1. One-dimensional virtual polytopes** are not various. A virtual segment is either a regular convex segment or an inverse to a convex segment. By Theorem 2.1, the latter is a polytopal function admitting the value $-1$ strictly inside a segment and admitting the value 0 outside it and at the endpoints.

**2. Two-dimensional virtual polytopes** are much more various. We list below all types of virtual triangles (i.e., virtual polytopes possessing 3 edges and therefore, 3 vertices).

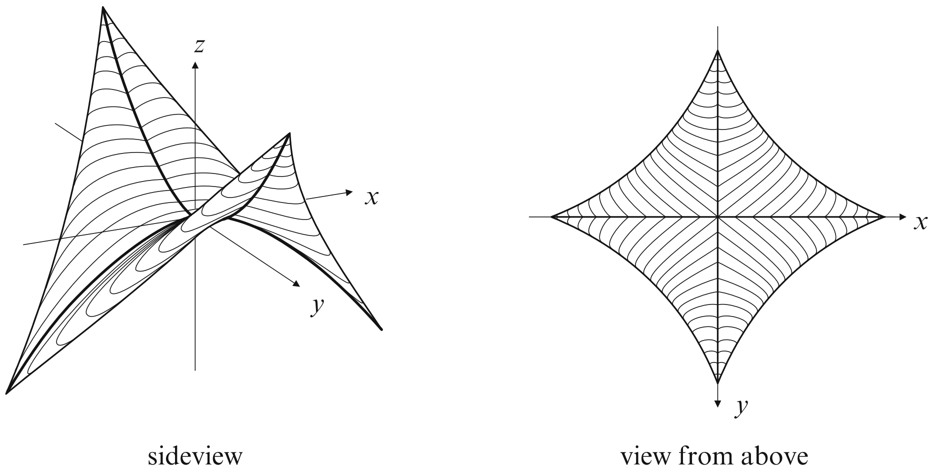



**Fig. 1**


**Fig. 2**

In the figure we indicate the values of the polytopal function and the coorientations of the edges. For instance, the second figure means that the polytopal function admits the value $-1$ strictly inside the triangle and inside the side edges. At the vertices and on the horizontal edge, the function equals 0.

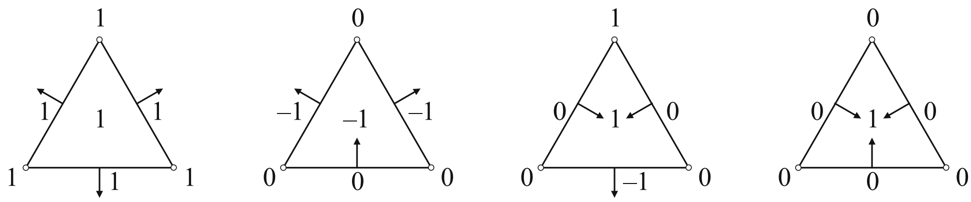



**Fig. 3**

The virtual polytope in Fig. 4 is not a triangle but a hexangle though the closure of its support is a triangle (similarly to the above virtual triangles). The point is that each of the three segments that serve as edges is taken twice with both coorientations.

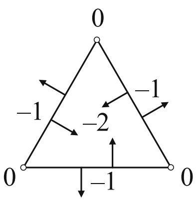



**Fig. 4**

3. **Some virtual tetrahedra. A hyperbolic tetrahedron.** It would take too much

space to list all 3-dimensional virtual tetrahedrons (i.e., virtual polytopes with 3 facets). Instead we draw some of them to further illustrate the theorems. This time we do not indicate the values of the polytopal functions. Instead, we show the coorientations of the facets by normal vectors and draw the fans (keeping in mind Theorem 2.10). It is possible to restore the values of the polytopal function owing the methods of [8].

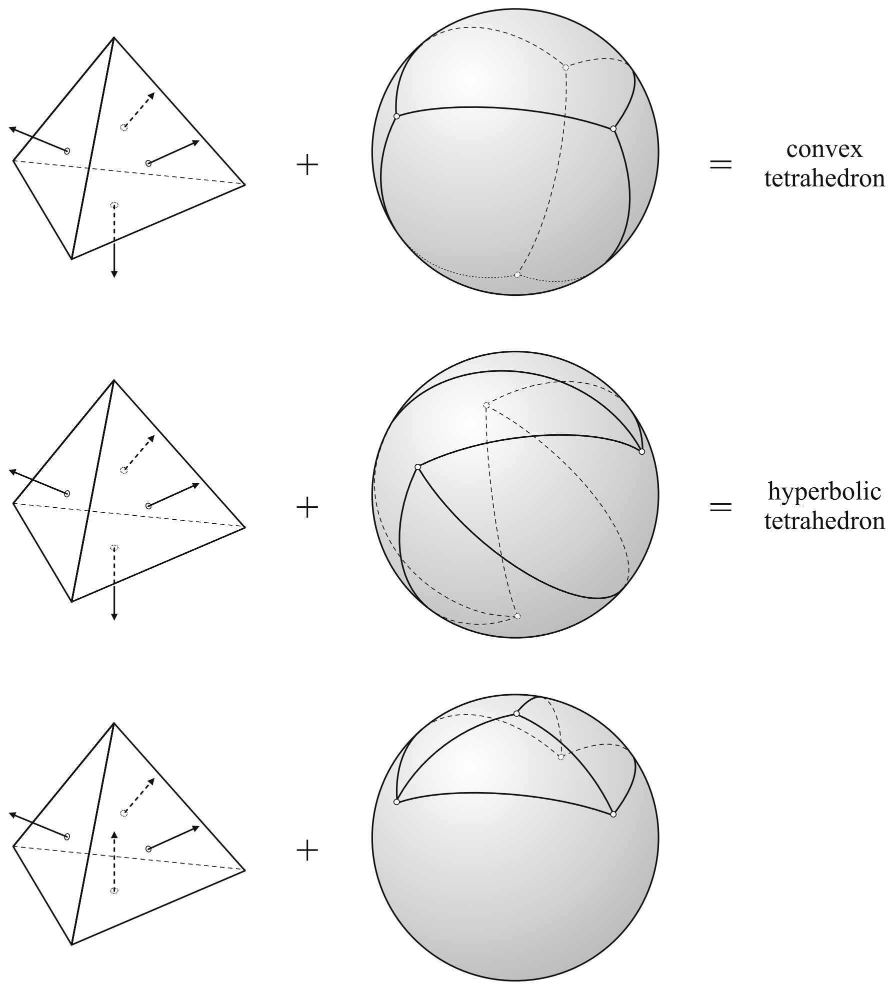



**Fig. 5**

The above *hyperbolic tetrahedron* is of particular interest. It is the simplest non-trivial example of a hyperbolic polytope.

## 4 Hyperbolic virtual polytopes: definition and properties

Let $K$ be a virtual polytope and let $h = h_K$ be its support function. For $\xi \in S^2$, let $e(\xi)$ be the plane defined by the equation $(x, \xi) = 1$. Consider the restriction of $h$ to the plane $e(\xi)$ and denote by $\mathcal{F} = \mathcal{F}_K(\xi)$ the graph of the restriction. The surface $\mathcal{F}$ is piecewise linear. Its vertices and edges correspond to those of $\Sigma_K$ intersected with the open hemisphere with the pole at $\xi$.

Note that the virtual polytope $K$ is convex if and only if the surface $\mathcal{F}_K(\xi)$ is convex for any $\xi$. This motivates the following definition.

**Definition 4.1.** A virtual polytope $K$ is called *hyperbolic* if $\mathcal{F}_K(\xi)$ is a saddle surface for any $\xi \in S^2$. In the sequel, we call such virtual polytopes simply *hyperbolic polytopes*.

Recall that a piecewise linear (or any other non-smooth surface) $F$ is called *a saddle surface* if there is no plane cutting a bounded connected component off $F$ (see [3]).

The polytopal complex $\mathbf{C}_K$ generated by a hyperbolic polytope $K$ is not a saddle surface (for there exists no closed saddle surface). Some of its vertices can be cut off the surface by a plane. Such vertices are called *horns* of the hyperbolic polytope $K$.

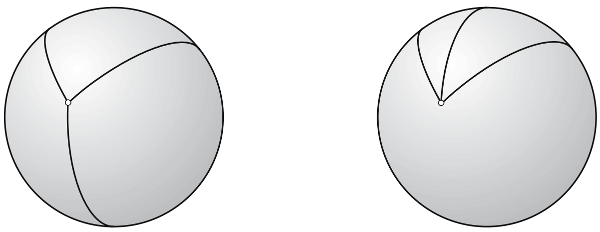
This preserves the traditional notation of the theory of smooth narrowing saddle surfaces, which deals with infinite horns (see [3] and [13]).

Let $\Xi = \{\xi_i\}$ be a collection of points on $S^2$ such that each open hemisphere contains at least one point from the collection.

**Proposition 4.2.** [9] *A virtual polytope $K$ is hyperbolic if and only if $\mathcal{F}_K(\xi)$ is a saddle surface for any $\xi \in \Xi$. In this case, $\mathcal{F}_K(\xi)$ is saddle for any other $\xi \in S^2$ as well.*

**Definition 4.3.** [9]

A vertex $\xi$ of a fan $\Sigma$ is *nonconvex* (respectively, *convex*) if there exists (respectively, doesn't exist) an adjacent to $\xi$ angle greater than $\pi$ (Fig. 6).



a convex vertex          a non-convex vertex

**Fig. 6**

Denote by $Hyp$ the set of all hyperbolic virtual polytopes. Also put
$Conv = \{K \in \mathcal{P}^* |$ either $K$ or $K^{-1}$ is a convex polytope$\}$.

The following theorem compares these sets and demonstrates their contraposition.

**Theorem 4.4.** *(1) $K \in Conv$ if and only if all non-boundary values of its facets are non-negative; $K \in Hyp$ if and only if all non-boundary values of its facets are non-positive.*

*(2) $K \in Conv \cup Hyp \quad \Leftrightarrow$*

*the coorientations of the facets of $K$ generate a global orientation of $\mathcal{C}_K$.*

*(3) Let $K$ be a simplicial virtual polytope (i.e., all its facets are virtual triangles). Then $K \in Conv$ if and only if every vertex of its fan $\Sigma_K$ is convex; $K \in Hyp$ if and only if every vertex of its fan $\Sigma_K$ is nonconvex.*

The above examples of virtual tetrahedrons (Fig. 5) give a good illustration of the assertions. To prove the theorem, we need two auxiliary lemmas.

**Lemma 4.5.** *Let $K$ be a 2-dimensional virtual polytope (embedded in $\mathbb{R}^3$). $K$ is hyperbolic if and only if all its values at non-boundary points are non-positive.*

**Proof.** The fan of $K$ is symmetric and has 2 vertices. Therefore, it suffices to consider the surface $\mathcal{F} = \mathcal{F}_K(\xi)$ for $\xi \perp aff(K)$. It is a piecewise linear cone. Let $e$ be a plane containing its apex $A$. By duality, $e$ corresponds to a point $E$ from the plane $k = aff(K)$. The point $E$ is a non-boundary point of $K$ if and only if the plane $e$ does not contain edges of $\mathcal{F}$.

The assertion of the lemma easily follows from the equality

$$K^{\xi}(E) = 1 + \chi(e \cap \overline{\mathcal{F}} \cap U(A)),$$

where $U(A) = \{x \in \mathbb{R}^3 \mid 0 \neq |x, A| < \varepsilon\}$ is a deleted neighbourhood of $A$, $\overline{\mathcal{F}}$ is the subgraph of the restriction of $h_K$ on the plane $e$, $\chi$ stands for the Euler characteristic.

Indeed, the surface $\mathcal{F}$ is saddle if and only if there exists a plane $e$ such that $e \cap \mathcal{F} = \{A\}$, which means $K(E) = 1$.

Now prove the equality.

$$-\chi(e \cap \overline{\mathcal{F}} \cap U(A)) =$$

by duality,

$$\#(\{l \mid l \subset k \textit{ is an oriented line, } E \in l;\ l \textit{ is a support line to } K\})/2 =$$

$$1 - K(E).$$

The latter equality is well-known for convex polytopes. Owing to linearity, it also is valid for virtual polytopes. □

**Lemma 4.6.** *Let $K$ be a virtual polytope. Suppose a point $X$ is a non-boundary point of its facet $K^{\xi}$. Suppose in addition that $X \notin aff(K^{\eta})$ for all $\eta \neq \xi$. Move somewhat the point $X$ in the direction $\xi$ (respectively, $-\xi$) and obtain the point $X^+$ (respectively, $X^-$).*

*In this notation, we have*

$$K^{\xi}(X) = K(M^-) - K(M^+).$$

**Proof.** The assertion follows directly from [8], Section 1. □

Now prove the theorem. The first assertion follows from Lemma 4.5. Indeed, for a virtual polytope $K$, the surface $\mathcal{F}_K(\xi)$ coincides in a neighbourhood of its vertex $A$ with the surface $\mathcal{F}_{K^A}(\xi)$, where $K^A$ is the facet of $K$ that corresponds to the vertex $A$.

It remains to observe that if $\mathcal{F}$ is non-saddle, there exists a plane cutting off a bounded component containing exactly one vertex of $\mathcal{F}$.

2. A virtual polytope $K$ can be regarded as a cycle (recall that $K$ is a polytopal function, i.e., a linear combination of convex polytopes). The winding number of the polytopal surface $\mathcal{C}_K$ at a point $X$ coincides with the value $K(X)$ provided that $X$ is a non-boundary point of $K$. Let $\eta$ denote the coorientation of $\mathcal{C}_K$ regarded as a cycle, whereas $\xi_i$ denote the orientations of the facets of $K$ (according to Definition 2.5). By Lemma 4.6, if $\eta$ coincides with (respectively, is opposite to) $\xi_i$, then all non-boundary values of $K^{\xi_i}$ are non-negative (respectively, non-positive).

3.The third assertion is trivial. □

## 5 Hyperbolic fans

In the section, all virtual polytopes (respectively, fans) are assumed to be simplicial (respectively, simple). Let $K$ be a hyperbolic polytope with a fan $\Sigma_K$ and a support function $h_K$. The edges of $\Sigma_K$ admit the following natural *coloring*: we color an edge of $\Sigma_K$ red (respectively, blue), if the graph $\mathcal{F}_K(\xi)$ of $h_K$ is concave up (respectively, concave down) in a neighbourhood of an inner point of the edge.

(We assume that the edge intersects with the hemisphere with the pole $\xi$.)

This construction is correct: the color does not depend on the choice of $\xi$.

**Remark 5.1.** For a hyperbolic $K$, the three edges adjacent to a vertex of $\Sigma_K$ can be colored only as is shown in Fig. 7.

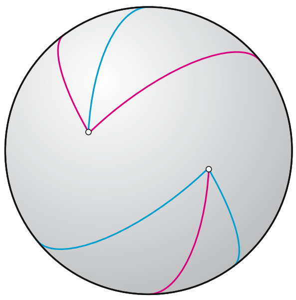


**Fig. 7**

**Definition 5.2.** A fan $\Sigma$ is *hyperbolic*, if each of its vertex is nonconvex.

A hyperbolic fan $\Sigma$ (which possibly is not a fan of a virtual polytope) is *proper*, if its edges admit a coloring such that the adjacent edges to every of its vertices are colored as in Fig. 7.

Given a hyperbolic fan, either there is no proper coloring or just two opposite ones. The below figure gives an example of a non-proper hyperbolic fan.

**Theorem 5.3.** *Let $K$ be a simplicial hyperbolic polytope with a fan $\Sigma_K$. For a 2-dimensional cell $\alpha$ of $\Sigma_K$, denote by $S(\alpha)$ the number of color changes as going around the boundary of the cell. The following assertions are valid:*

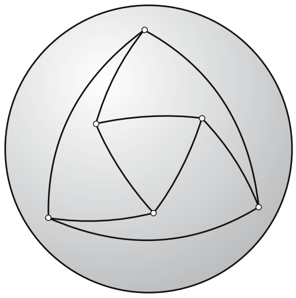


**Fig. 8**

1. $S(\alpha) = 2$ *if and only if* $\alpha$ *corresponds by duality to a horn.*
2. $S(\alpha) = 2$ *implies that* $\alpha$ *contains a great semicircle (not vise versa!).*
3. $S(\alpha) = 0$ *implies that* $K$ *is a virtual triangle or a virtual segment.*
4. $\sum_\alpha [S(\alpha) - 4] = -8$.

**Proof.** Denote by $h$ the support function of $K$.

Prove 1 and 2. Let $A$ be a horn of the virtual polytope $K$. Let a Cartesian coordinate system $(x, y, z)$ be chosen so that the vertex $A = (0, 0, 0)$ of the virtual polytope $K$ can be cut off by the plane $z = -\epsilon$ for a small $\epsilon > 0$.

This means (by Remark 2.11) that $h'_x = h'_y = h'_z = 0$ on the cell $\alpha$, whereas $h'_z \leq 0$ on neighbour to $\alpha$ cells.

Consider the restriction of $h$ to a plane $e(\xi)$ such that the $z$ axis is orthogonal to $\xi$.

Let $\overline{\alpha} = \overline{\alpha} \subset e(\xi)$ be the polygon which corresponds to the cell $\alpha$ (i.e., the intersection of $e(\xi)$ with the cone built over $\alpha$. For a vertex $X$ of $\overline{\alpha}$, three cases are possible (see Fig. 9):

**a.** Two red edges adjacent to $X$ admit a one-sheet orthogonal projection on $z^\perp$. The polygon $\overline{\alpha}$ lies beneath the red edges.

**b.** Two blue edges adjacent to $X$ admit a one-sheet orthogonal projection on $z^\perp$. The polygon $\overline{\alpha}$ lies over the blue edges.

**c.** Locally, $\overline{\alpha}$ lies between a blue and a red edges that have the same projection on $z^\perp$.

Assume that $dim K = 3$. Altering the vector $\xi$, make the plane $e(\xi)$ contain a vertex $X$ of $\overline{\alpha}$ of the type c. Let $\overline{s}$ be a ray with the apex at the point $X$ which locally lies in $\overline{\alpha}$. Then $\overline{s}$ is contained in $\overline{\alpha}$.

Treating similarly other planes $e(\xi)$ (such that $\xi$ is orthogonal to $z$), we conclude that the color changes twice as going around $\alpha$ and that $\alpha$ contains a great semicircle.

Alternatively, let $K$ be a hyperbolic polytope and let the color change twice as going around a cell $\alpha$ of $\Sigma_K$. Let $\alpha$ correspond by duality to a vertex $A$. For the above choice of coordinate system, the polytope $\overline{\alpha}$ looks locally as shown in Fig. 9. In each of these three cases, we have $h'_z \leq 0$ in a neighbourhood of the cell $\alpha$. Since $h'_z = 0$ on $\alpha$, there exists a plane cutting a bounded component with the vertex $A$ off the complex $\mathbf{C}_K$, i.e., $A$ is a horn.

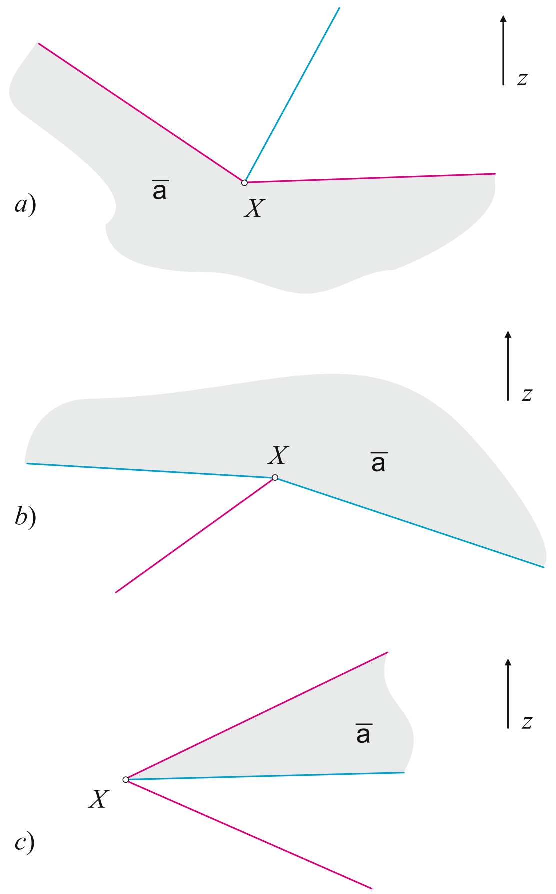



**Fig. 9**

3. Suppose $S(\alpha) = 0$. Then each angle of the spherical polygon $\alpha$ is greater than $\pi$. This means that either $\alpha$ is a spherical 2-gons or $S^2 \setminus \alpha$ is contained in an open hemisphere, which is impossible.

4.We follow the proof of the Cauchy Lemma (see [2]). Denote by $V$ and $F$ the number of vertices and the number of 2-dimensional cells of $\Sigma_K$. Count the total number of color changes for all cells. Since each vertex gives exactly two changes, we have $\sum_\alpha S(\alpha) = 2V$. Applying the Euler formula $2F = V + 4$, we conclude the proof. □

**Definition 5.4.** We say that two proper hyperbolic fans are *combinatorially equivalent* if there exist proper colorings of both of them together with a combinatorial equivalence that preserves the colors of the edges.

**Definition 5.5.** We say that a hyperbolic fan $\Sigma$ is *realizable* (respectively, *combinatorially realizable*) if there exists a hyperbolic polytope $K$ such that $\Sigma_K = \Sigma$ (respectively, $\Sigma$ is combinatorially equivalent to $\Sigma_K$).

Given one realizable fan, we sometimes can obtain many others. The operations described below change the combinatorics of a fan but preserve combinatorial realizability.

**Proposition 5.6.** *Let $\Sigma$ be a realizable hyperbolic fan. Suppose two inner points of some* *red (respectively, blue) edges can be connected by a geodesic segment avoiding intersections with other edges. Then breaking somewhat these red (respectively, blue) edges, coloring this segment blue (respectively, red), and adding it to $\Sigma$, we obtain a combinatorially realizable hyperbolic fan. This is called $\mathcal{H}$-operation.*

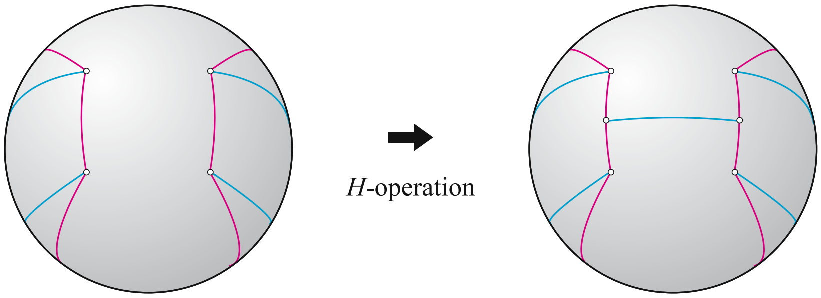



**Fig. 10**

**Proof.** Let a virtual polytope $K$ correspond to the fan $\Sigma$. For a fixed vector $\xi$, consider the surface $\mathcal{F}_K(\xi)$. Let the new segment $s$ belong to a cell $\alpha$, which corresponds to the facet $A$ of the surface $\mathcal{F}_K(\xi)$. Put $a = \text{aff}(A)$. Now break the plane $a$ along the segment $s$ to obtain a concave down (respectively, concave up) surface. Owing to simplicity of the fan, replacment of $a$ by the broken plane causes no harm to the combinatorics at adjacent vertices. This gives a saddle surface with the desired combinatorics. Alter the surfaces $\mathcal{F}_K(\xi)$ for other vectors $\xi$ to get a self-consistent collection of surfaces. This corresponds to a hyperbolic polytope of the desired type. □

Not all hyperbolic polytopes allow an $\mathcal{H}$-operation. For instance, the hyperbolic tetrahedron (Fig. 5) allows no $\mathcal{H}$-operation.

We shall apply $\mathcal{H}$-operations in Section 6 for shortening the edges of hyperbolic fans, which is necessary for further smoothing.

**Theorem 5.7.** *Let $K$ be a hyperbolic polytope. Suppose that one of the below described $\mathcal{C}$- or $\mathcal{S}$- configurations of four geodesic segments $1, 2, 3,$ and $4$ can be placed on $S^2$ (see Fig. 11) such that*

- *The endpoints of 1 and 4 of $\mathcal{C}$-configuration (respectively, $\mathcal{S}$-configuration) lie on edges of $\Sigma_K$ of different (respectively, one and the same )color.*
- *Intersections of the configuration with the edges of $\Sigma_K$ are avoided (except for the endpoints of 1 and 4).*
- *Segments 2 and 3 are great semicircles.*
- *The vertices of the configuration are nonconvex.*
- *Segments 1 and 4 lie on one and the same great circle.*

*Then after an appropriate coloring and adding this configuration to $\Sigma_K$, we obtain a combinatorially realizable fan. This is called $\mathcal{C}$-operation (respectively, $\mathcal{S}$-operation).*

**Proof.** Consider the surface $\mathcal{F}_K(\xi)$ for a fixed vector $\xi$. Denote by $a$ its face corresponding to the cell of $\Sigma_K$ which contains the $\mathcal{S}$-configuration (respectively, $\mathcal{C}$-configuration).

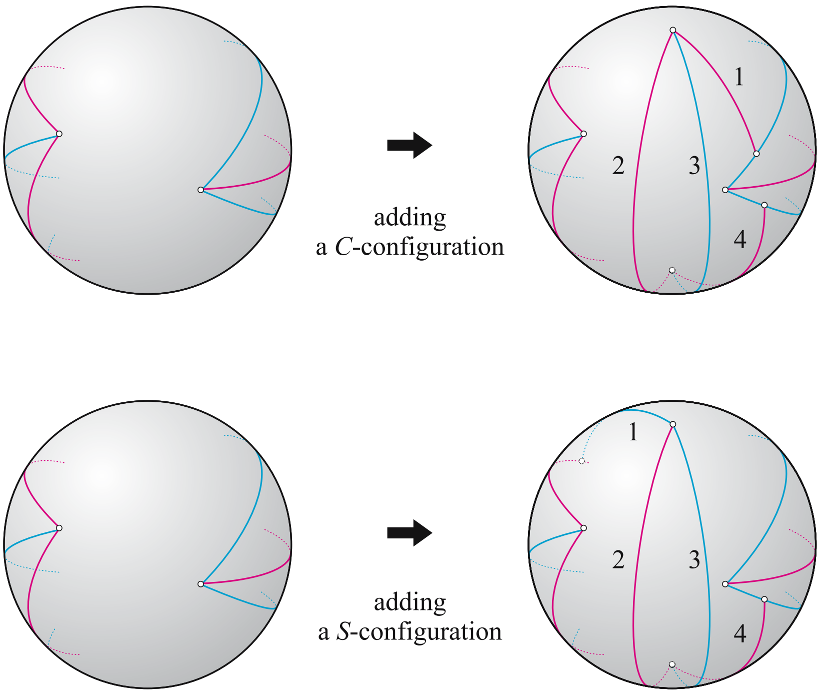



**Fig. 11**

Replace the affine hull of $a$ by a polyhedral surface (consisting of three linear parts) as is shown in Fig. 12. The affine hulls of other faces remain unchanged.

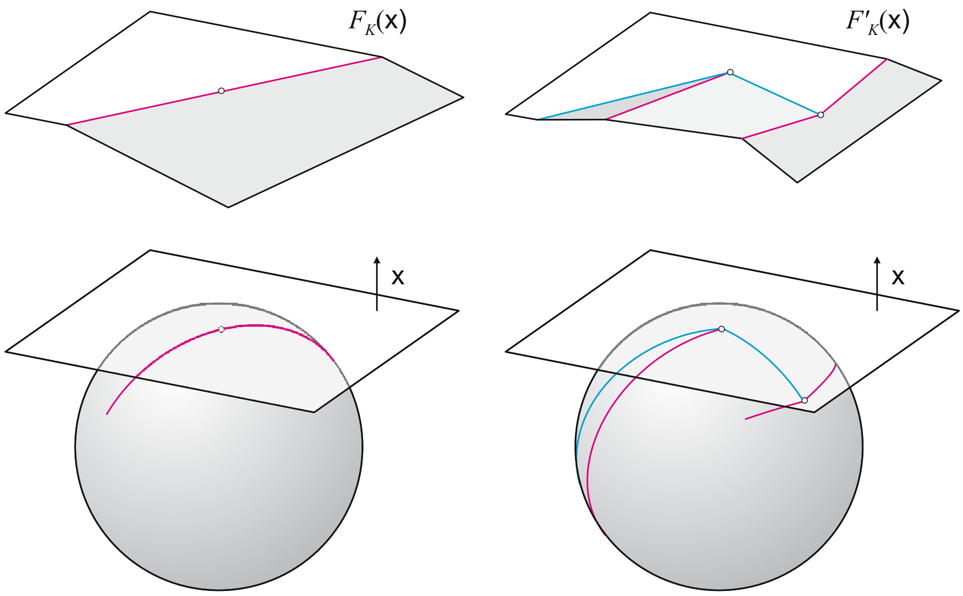



**Fig. 12**

Thus we obtain a new saddle surface $\mathcal{F}'_K(\xi)$ for the fixed vector $\xi$. Altering the surfaces $\mathcal{F}_K(\xi)$ for other planes $\xi$, we obtain a self-consistent collection of saddle surfaces, and therefore, a hyperbolic polytope of the desired type. $\square$

An easy calculation of color changes (according to Theorem 5.3) proves the following proposition.

**Proposition 5.8.**

- *For a 3-dimensional hyperbolic polytope, $\mathcal{H}$-operations never change the number of horns.*
- *$\mathcal{C}$-operation (respectively, $\mathcal{S}$-operation) doesn't change the number of horns if the $\mathcal{C}$-configuration (respectively, $\mathcal{S}$-configuration) is contained in a cell corresponding to a horn.*
- *Otherwise , $\mathcal{C}$-operation (or $\mathcal{S}$-operation) adds one horn.*

It is often impossible to add a horn by a $\mathcal{C}$- or a $\mathcal{S}$ -operation to a hyperbolic polytope $K$. Indeed, there must exist a cell of $\Sigma_K$ containing a great semicircle but not corresponding to a horn of $K$. For instance, neither the hyperbolic polytope from [5], nor those from [9] possess this property.

## 6 Advanced examples: hyperbolic polytopes with even and odd number of horns

Let $K$ be a simplicial hyperbolic polytope. Let $\alpha_1, ...\alpha_N$ be the cells of $\Sigma_K$ such that for each $k$, the color changes twice as going around the boundary of $\alpha_k$. (Recall that each such cell corresponds by duality to a horn.) By Theorem 5.3, each cell $\alpha_k$ contains an oriented great semicircle, say, $S_k$. Its orientation is generated by the coloring. Arrangements of great semicircles on $S^2$ may have a complicated structure. Their combinatorial classification is a separate problem (not to be discussed here). In this paper, we bound ourselves by simple combinatorics of $\{S_i\}_{i=1}^N$.

**Horns ordering property.** Let $K$ be a hyperbolic polytope.

$K$ is said to possess the *horns ordering property*, if. there exists a hemisphere $S^2_+$ and a circular ordering of the set $\{S_i\}_{i=1}^N$ (and therefore, the same ordering of $\{\alpha_i\}_{i=1}^N$ ) satisfying the following two conditions.

1. For each $i$, the prolongation of the great semicircle $S_i$ in $S^2_+$ first meets the semicircle $S_{i+1}$.

2. For each $i$, the prolongation of the great semicircle $S_i$ in $S^2 \setminus S^2_+$ first meets the semicircle $S_{i-1}$.

In the section, all hyperbolic polytopes constructed possess this property.

Denote by $S$ the boundary of $S^2_+$. Choose an orientation of $S$ and mark the cells $\alpha_k$ by a sign " $+$ " or " $-$ " as follows: if $S$ first meets a red edge of $\alpha_K$, then assign to $\alpha_k$ the sign " $+$ ". Otherwise, we assign the sign " $-$ " (see Fig. 14).

Thus we obtain a *necklace* (i.e., a circular sequence) of $N$ signs $\mathcal{N}(K) = (\pm\pm ...\pm)$ which is defined up to the order inversion combined with the sign inversion (this combination is motivated by the orientation changing of $S$).

The example from [5] gives the necklace $(+ - +-)$. The examples from [9] give the necklace of type $(+ - + - ... + -)$ for even number $N \geq 4$ of signs.

However, the set of realizable necklaces is much more diverse:

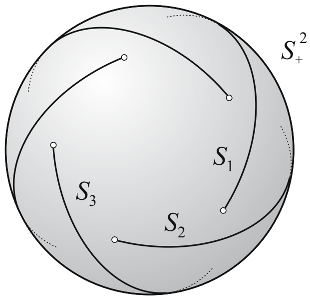



**Fig. 13**

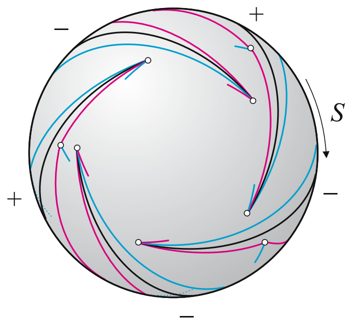



**Fig. 14**

**Theorem 6.1.** *Suppose the sign changes at least 4 times as going around a necklace* $\mathcal{N} = (\pm\pm\ldots\pm)$. *Then there exists a hyperbolic virtual polytope* $K$ *such that* $\mathcal{N}(K) = \mathcal{N}$.

**Proof. Step 1. New hyperbolic polytope of type** $(+-+-)$**.**

Assuming that a coordinate system $(x, y, z)$ is fixed, consider the collection of points $\{A_i, P_i, O\}_{i=1}^4$ as shown in Fig. 15.

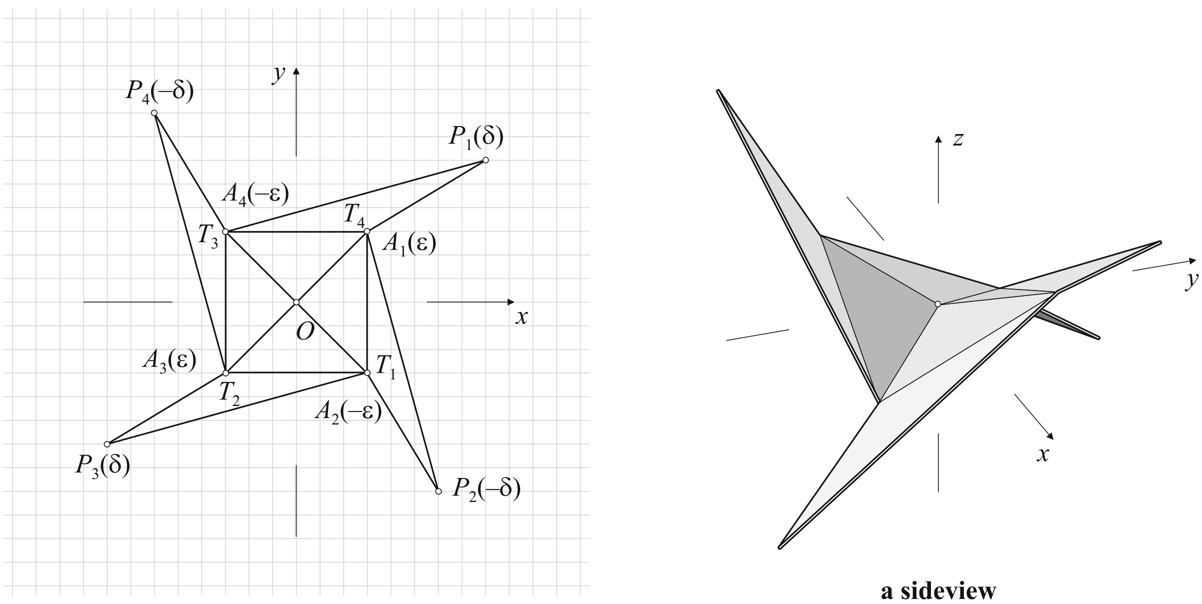



**Fig. 15**

The $x$ and $y$ coordinates of the points can be read from the grid, whereas the $z$ coordinates are indicated in the brackets. For instance, $A_1 = (3, 3, \epsilon)$. The values $\delta > 0$ and $\epsilon > 0$ are chosen below.

The collection of oriented triangles

$$S_i = (A_i A_{i+1} O), \quad T_i = (A_{i+1} A_i P_i),$$

$$S_i' = (A_{i+1} A_i O), \quad T_i' = (A_i A_{i+1} P_i)$$

forms a simplicial complex **C**. The pairs of triangles $S_i$ and $S_I'$ (as well as $T_i$ and $T_i'$) differ only by orientation: the normal vectors of $S_i$ and $T_i$ look upwards (i.e., form an angle with the $z$ axis less than $\pi/2$), whereas the normal vectors of $S_i'$ and $T_i'$ look downwards (i.e., form an angle with the $z$ axis greater than $\pi/2$).

**Remark 6.2.** We indicate the orientation of a triangle in two ways: by its normal vector $\xi$ and by the order of its vertices.

**Remark 6.3.** Here and in the sequel, given an ordered set of any objects $\{X_i\}_{i=1}^M$, we assume that for any $k \in \mathbb{Z}$, we have $X_k = X_i$ if $k \equiv i (mod M)$.

Keeping in mind Theorem 2.10, mark on the sphere $S^2$ the endpoints of the normal vectors of the triangles (we denote them by the same letters). For an appropriate choice of the numbers $\delta$ and $\epsilon$ (for instance, the values $\epsilon = 0,1$ and $\delta = 0,4$ are suitable), the geodesic segments connecting the points (see Fig. 16) do not intersect each other (except for the marked points). The complex **C** together with the fan obtained gives a hyperbolic polytope $K_4$.

To adjust it for further smoothing (Section 7), apply four $\mathcal{H}$-operations and obtain the fan $\Sigma_4'$ (Fig. 16). By Theorems 4.4 and 5.3, this yields a hyperbolic polytope $K_4'$ with 4 horns (namely, the points $P_i$), which differs from that constructed in [5]: their fans are not combinatorially equivalent although they contain the same number of vertices and edges.

**Step 2. New hyperbolic polytopes of type $(+ - ... + -)$ with even number of horns.**

On this step, we take a star with $N$ vertices (as defined in [9]) instead of the four points $A_1, ..., A_4$ and follow the pattern of Step 1. Again, the $z$ coordinates of the points are indicated in the brackets.

The collection of oriented triangles

$$S_i = (A_i A_{i+1} O), \quad T_i = (A_{i+1} A_i P_i),$$

$$S_i' = (A_{i+1} A_i O), \quad T_i' = (A_i A_{i+1} P_i)$$

form a simplicial complex **C**. As on the Step 1, the pairs of triangles $S_i$ and $S_I'$ (as well as $T_i$ and $T_i'$) differ only by orientation: the normal vectors of $S_i$ and $T_i$ look upwards, whereas the normal vectors of $S_i'$ and $T_i'$ look downwards.

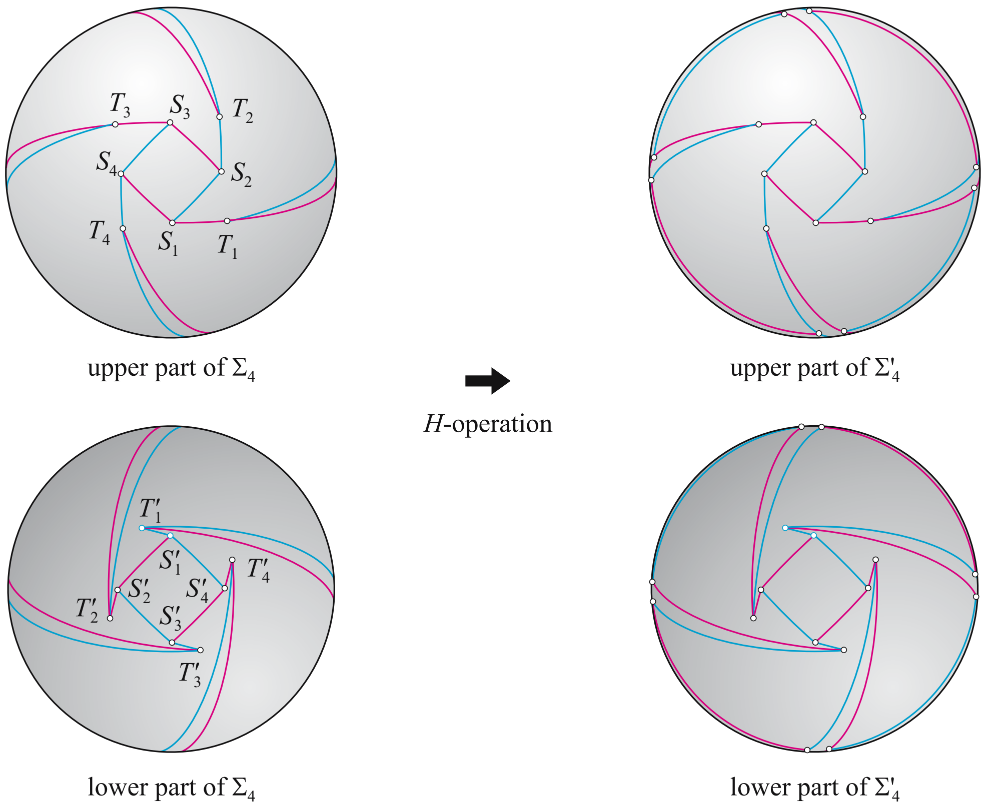



**Fig. 16**

As above,we mark on the sphere $S^2$ the endpoints of the normal vectors of the triangles (we denote them by the same letters). For an appropriate choice of the numbers $\delta$ and $\epsilon$, the geodesic segments connecting the points (see Fig. 17) do not intersect each other (except for the marked points). The complex **C** together with the fan obtained $\Sigma_N$ gives a hyperbolic polytope $K_N$.

After a series of $\mathcal{H}$-operations, we obtain a hyperbolic virtual polytope $K'_N$ with $N$ horns (namely, the points $P_i$), which differs combinatorially from that constructed in [8]. (Although it contains the same number of vertices and edges). An advantage of such a polytope is that it admits addition of extra horns.

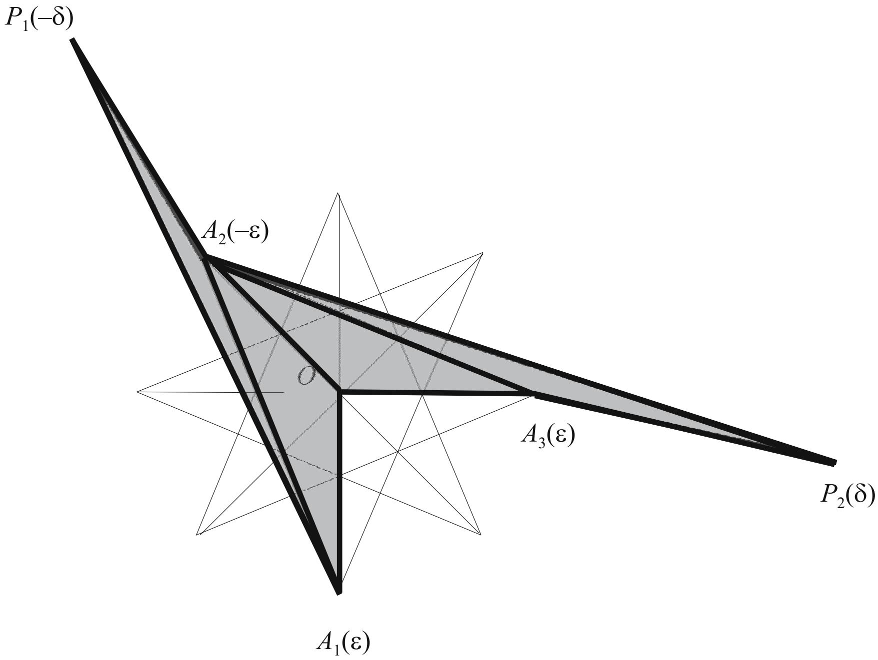
**Step 3. Adding an extra horn. A hyperbolic polytope of type** $(++-+-...+-)$ **with odd number of horns.**

The fan of the above constructed hyperbolic polytope $K_N$ allows a $\mathcal{S}$-operation (see Fig. 18). Thus we obtain a desired hyperbolic polytope.

This time the necessary conditions for further smoothing $\mathcal{H}$-operations are a bit more complicated than those on the previous steps. Nevertheless, they lead to a smoothable hyperbolic polytope with $N+1$ horns.

**Step 4. Arbitrary necklace** $\mathcal{N} = (\pm\pm...\pm)$**.**

First construct a hyperbolic polytope of type $(+,-,...,+,-)$ with $N$ horns, where $N$ equals the number of color changes in the necklace $\mathcal{N}$. Applying necessary $\mathcal{S}$-operations, we insert additional horns. After a series of $\mathcal{H}$-operations (as on Step 3), we obtain the



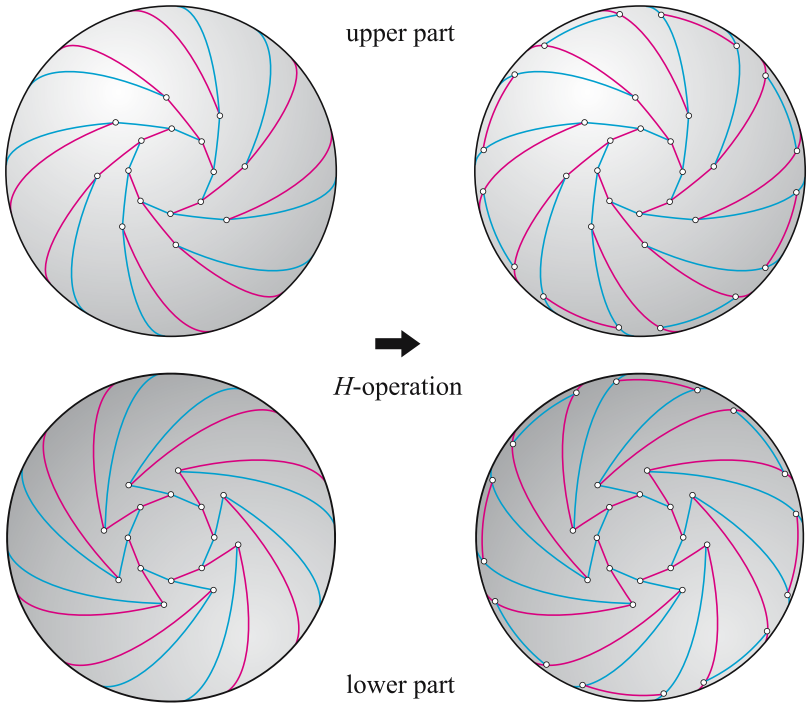




**Fig. 17**

required. □

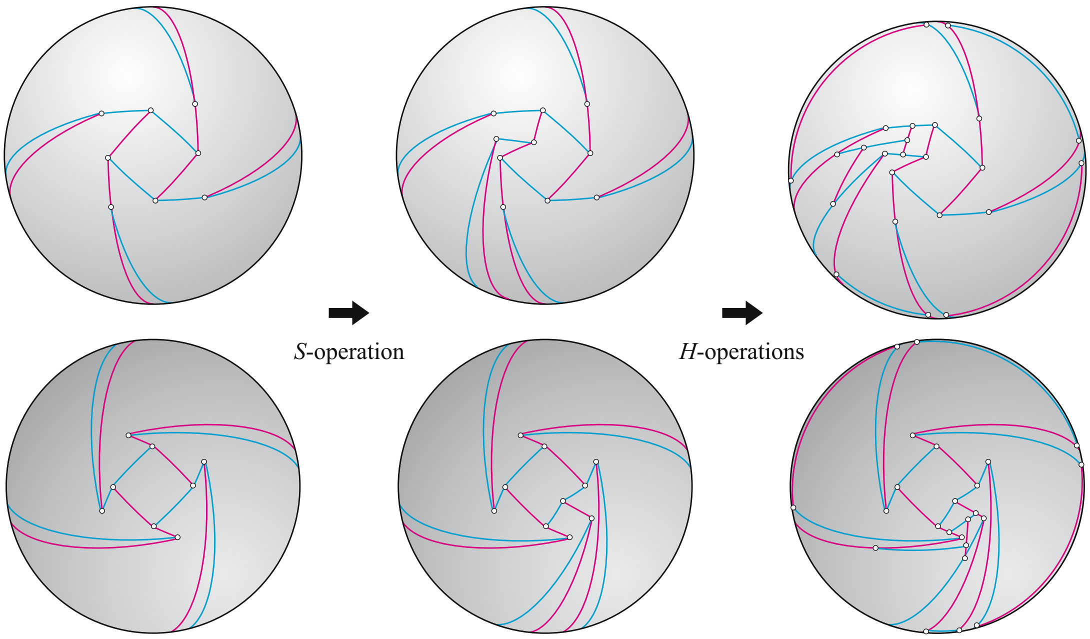



**Fig. 18**

Using the polytopes $K_N$ as a base for applying $\mathcal{S}$- and $\mathcal{C}$-operations, one has even more freedom. This technique leads to advanced (in comparison with constructed above) combinatorial types of hyperbolic polytopes. However, we leave them for a later paper.

## 7 Smoothing techniques

Recall that whenever we have a smooth hyperbolic approximation of a hyperbolic virtual polytope, we obtain a counterexample to A.D. Alexandrov's hypothesis.

**Theorem 7.1.** *Each polytope constructed in Section 6 admits a hyperbolic $C^\infty$-smoothing. In particular, there exist $C^\infty$-smooth hyperbolic hérissons with odd number of horns.*

**Proof.** We follow the pattern of [9]: to obtain the desired approximation, we find mutually consistent smooth saddle surfaces which approximate the surfaces $\mathcal{F}_K(\xi)$ for different $\xi$. The approximating smooth surface $\mathcal{F}'_K(\xi)$ coincides with $\mathcal{F}_K(\xi)$ at the points lying far from the edges. Along the edges, the surface $\mathcal{F}_K(\xi)$ is replaced by cylinders and cones.

By a *cylinder* (respectively, by a *cone* with a vertex $A$) we mean a set of points that is invariant under all translations parallel to a line $l$ (respectively, under homotheties with center in $A$).

Note that as passing to another vector $\xi$, cones and cylinders may turn to each other. That is why we need different types of local saddle approximations as given in Lemma 7.2.

**Lemma 7.2.** *Let $F$ be a piecewise linear surface given in the coordinate system* $(u, v, w)$

*by the formula*

$$w(u,v) = \begin{cases} 0 & \text{if } v < |u|, \\ |u| - v & \text{otherwise.} \end{cases}$$

*Let $L_1, L_2$, and $L_3$ be its edges (see Fig. 19); let $A_i$ be a point lying on $aff(L_i) \setminus L_i$ or the point lying on $aff(L_i)$ "at infinity".*

*Then $F$ admits a $C^\infty$-smooth approximation by a saddle surface $F'$ such that*

- *The surface $F'$ coincides with $F$ far from the edges.*
- *Along the edges $L_i, (i = 1, 2, 3)$, but far from the vertex $O$, the surface $F'$ is a cone with the apex at $A_i$ (i.e., a cylinder if $A_i$ lies at infinity).*
- *The cylinders (or the cones) approximating $L_2$ and $L_1$ can be chosen independently.*

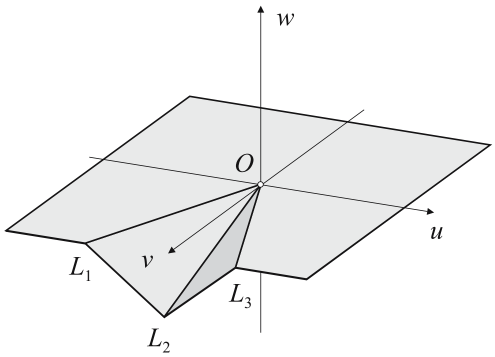



**Fig. 19**

**Proof.** The following two cases are already proven in [9]:

1. $L_1, L_2$, and $L_3$ are approximated by cylinders.

2. $L_1$ and $L_3$ are approximated by cones with the apex at $O$, whereas $L_2$ is approximated by a cylinder.

Clearly case 2 is the tightest and implies the whole lemma. □

Continue the proof of the theorem.

**Step 1.** Consider a narrow belt about the equator $z = 0$ on the sphere $S^2$ and construct first a local approximation in the belt.

The fan of $K$ intersected with the belt splits into disjoint union of figures of two types (see Fig. 20).

Owing to $\mathcal{H}$-operations, we may assume that all edges of the fan $\Sigma_K$ are shorter than $\pi/2$. Consider a collection of hemispheres with poles in some points $\{\eta_i\}$ lying on the equator such that the union of the hemispheres covers the belt. Choose mutually consistent approximations of the surfaces $\mathcal{F}(\eta_i)$ along the belt as follows:

- Edges of type 1 coming out of of the belt, are approximated by cones with the apex at the vertex corresponding to $A$ (i.e., at the vertex lying on the equator $z = 0$.
- Edges of type 2 coming out of the belt, are approximated by cylinders.

**Step 2.** Put $\xi = (0, 0, \pm 1)$ and consider the surface $\mathcal{F}_K(\xi)$. The surface has already some approximations (arising from Step1) along the edges coming from infinity. Namely,

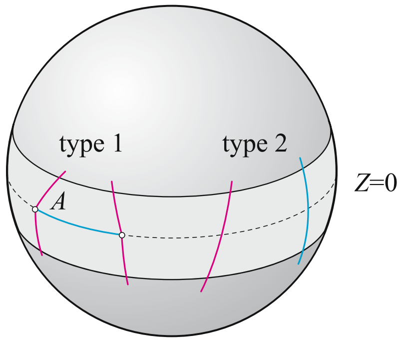



**Fig. 20**

if an edge arises from type 1, it is approximated by a cylinder. If an edge comes from type 2, it is approximated by a cone with an apex lying beyond the edge. The only (mild) condition on the approximations we choose is that the cones and cylinders must be narrow (i.e., $\mathcal{F}'$ differs from $\mathcal{F}_K(\xi)$ in a narrow domain along the edges).

**Step 3.** Using the freedom of choice (Lemma 7.2), one can expand these approximations to a global saddle surface $\mathcal{F}'$, approximating $\mathcal{F}_K(\xi)$. □

## Acknowledgment

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