ISOTOPY PROBLEMS FOR SADDLE SURFACES.

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Abstract. Four mutually dependent facts are proven.

- A smooth saddle sphere in S^3 has at least four inflection arches.
- Each hyperbolic hérisson H generates an arrangement of disjoint oriented great semicircles on the unit sphere S^2. On the one hand, the semicircles correspond to the horns of the hérisson. On the other hand, they correspond to the inflection arches of the graph of the support function h_H.

  The arrangement contains at least one of two basic 4-arrangements.
- A new type of a hyperbolic polytope with 4 horns is constructed.
- There exist two non-isotopic smooth hérissons with 4 horns.

This is important because of the obvious relationship with extrinsic geometry problems of saddle surfaces, and because of the non-obvious relationship with A.D. Alexandrov's uniqueness conjecture.

1. Introduction

The paper proceeds the study of hyperbolic virtual polytopes, hyperbolic hérissons, and associated saddle surfaces. These notions arose originally as a tool for constructing counterexamples to the following uniqueness conjecture, proven by A.D. Alexandrov (see [1]) for analytic surfaces.

Uniqueness conjecture for smooth convex surfaces.
Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

Given a counterexample K to the conjecture, the Minkowski difference of K and the ball of radius C is a hyperbolic hérisson. Conversely, the Minkowski sum of a hyperbolic hérisson and a sufficiently large ball is a counterexample to the conjecture (see [6]).

With a hyperbolic hérisson we associate the dual object, namely, the spherical graph of its support function (Section 2). It is a sphere-homeomorphic closed (spherically) saddle surface embedded in the sphere S^3.

Theorem 1.1. *(see Theorem 3.2.) Let Γ be a two-dimensional closed smooth saddle surface lying in S^3 and admitting the bijective orthogonal projection on some great sphere $S^2 \subset S^3$. Assume that Γ is non-degenerate, i.e., it does*

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not coincide with a great sphere. Then Γ has at least 4 inflection arches (see Definition 3.1). □

To prove the theorem, we use the technique developed by A.V. Pogorelov in [11]. In the paper, he erroneously asserts that the above A.D. Alexanrov's conjecture is true. We indicate his gap and demonstrate that his method leads directly to the Theorem 3.2.

Corollary 3.4. *Each hyperbolic herisson H generates an arrangement of at least four disjoint oriented great semicircles on the unite sphere S^2. There is a natural one-to-one correspondence*

"semicircles of the arrangement $\leftrightarrow$ horns of the herisson $\leftrightarrow$ inflection arches of the graph of the support function h_H". □

The following theorem is a discrete version of the Theorem 3.2.

Theorem 3.5. *Let Γ be a non-degenerate two-dimensional closed polytopal saddle surface lying in S^3 (i.e., all the facets of Γ are some spherical polygons). Suppose that Γ admits the bijective orthogonal projection on some great sphere $S^2 \subset S^3$. Then Γ contains at least 4 disjoint facets $s_1, s_2, ..., s_k$ such that*

(1) *each of s_i is bounded by two convex broken lines (say, by L_1 and L_2, see Fig.7)*;
(2) *each s_i contains a great semicircle*;
(3) *the surface Γ is concave up along one of the broken lines L_1 and L_2. It is concave down along the other broken line.* □

Definition 1.2. Two smooth hérissons H_0 and H_1, both with 4 horns, are called *isotopic* if there exists a continuous family of hérissons H_t which starts at H_0 and ends at H_1 such that for any $t \in [0, 1]$, the hérisson H_t has exactly four horns.

Example 4.2. *We show that there exist two non-isotopic arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of four great semicircles (see Fig. 8). After that, we construct a hyperbolic hérisson with 4 horns which generates the arrangement $\mathfrak{A}_2$. The already known example by Martinez-Maure (see Fig. 1) generates the arrangement $\mathfrak{A}_1$. Since the arrangements are non-isotopic, the hérissons are non-isotopic as well.* □

To construct the new hyperbolic hérisson, we first construct a polytopal saddle surface spanned by some special linkage on the 3-dimensional sphere. This yields a new hyperbolic virtual polytope with 4 horns. Then we apply the smoothing technique and get the required hérissons.

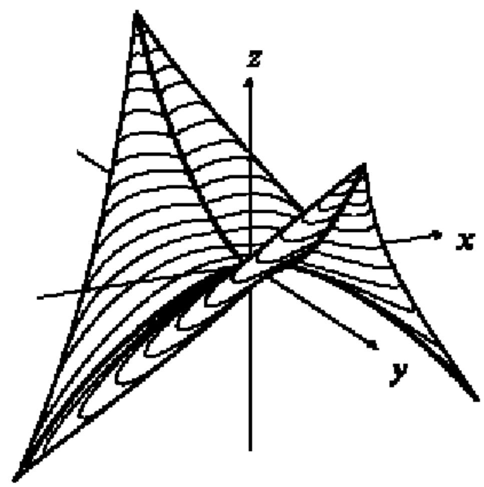

Figure 1

2. Preliminaries

The theory of virtual polytopes ([4], [8], [10]) and the theory of hérissons ([6], [7], [12]) gives a geometric interpretation of the Minkowski difference of convex polytopes and smooth convex bodies.

Here we sketch briefly the part of this theory to be used in the paper, referring the reader to [8], [9], and [10] for details.

Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a continuous positively homogeneous function which is either piecewise linear or C^2-smooth.

If h is a convex function, then it is the support function of a convex polytope or of a smooth convex body.

Two cases are of particular interest: when h is (at least C^2) smooth and when h is piecewise linear. In both cases, h is the difference of two convex functions, either piecewise linear or smooth, so it makes sense to interpret h as the support function of the Minkowski difference of the corresponding objects (either smooth bodies or polytopes).

We associate below with such a function h two mutually dual objects: a surface in $\mathbb{R}^3$, which generalizes the correspondence "support function $\leftrightarrow$ convex body", and the spherical graph of h.

Hérissons as surfaces in $\mathbb{R}^3$. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a continuous positively homogeneous C^2-smooth function. By the *hérisson H with the support function h* we mean the envelope of the family of planes $\{e_h(\xi)\}_{\xi \in S^2}$, where $e_h(\xi)$ is given by the equation $(\xi, x) = h(\xi)$. It is a sphere homeomorphic surface with possible self-intersections and self-overlappings.

As a set of points, a hérisson H coincides with the image of the mapping

$$\varphi : S^2 \longrightarrow \mathbb{R}^3,$$

$$(x, y, z) \longrightarrow (h'_x(x, y, z), h'_y(x, y, z), h'_z(x, y, z)).$$

If we start with a convex function h, then H is known to be the boundary of the convex body with the support function h.

Virtual polytopes as surfaces in $\mathbb{R}^3$. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a continuous positively homogeneous piecewise linear function.

The *fan* Σ_h of the function h is defined as the minimal tiling of $\mathbb{R}^3$ such that h is linear on each tile. It consists of some cones with a common apex at O. We shall depict the *spherical fan*, i.e., the intersection of Σ_h with the unit sphere centered at O.

It is possible to associate with h some polytopal surface H (see [7-10]) which is called the *virtual polytope with the support function h.* The surface H is combinatorially dual to the fan Σ_h, and the coordinates of the vertices can be easily read off from the function h.

Namely, each 3-dimensional tile σ of Σ_h corresponds to the vertex of H with coordinates

$$((h\mid_\sigma)'_x(\cdot), (h\mid_\sigma)'_y(\cdot), (h\mid_\sigma)'_z(\cdot)).$$

Here $h\mid_\sigma$ stands for the restriction of h on the tile σ. Since it is a linear function, the expression does not depend on the point of the tile.

This repeats literally the way of reconstruction of a convex polytope by its support function.

Spherical graph. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a positively homogeneous continuous function. It makes sense to draw its graph on the 3-dimensional sphere. Fix an embedding of the 3-dimensional real space $\mathbb{R}^3$ in $\mathbb{R}^4$. The unit sphere centered at O in $\mathbb{R}^3$ (respectively, in $\mathbb{R}^4$) is denoted by S^2 (respectively, by S^3). Denote by Γ its graph. The intersection of Γ with the sphere S^3

$$\Gamma_{sph}(h) = \Gamma(h) \cap S^3$$

is called the *spherical graph* of the function h.

It is a closed 2-dimensional surface. The spherical central projection

$$\pi : S^3 \setminus \{(0, 0, 0, 1), (0, 0, 0, -1)\} \to S^2$$

maps $\Gamma_{sph}(h)$ one-to-one to S^2 (see Fig. 2).

Definition 2.1. A surface $F \subset \mathbb{R}^3$ is called *a saddle surface* if there is no plane cutting a bounded connected component off F.

Equivalently, a surface F is *saddle* if no plane intersects F locally at just one point.

Analogously, a surface $F \subset S^3$ is called *a spherically saddle surface* if no great 2-dimensional sphere intersects F locally at just one point.

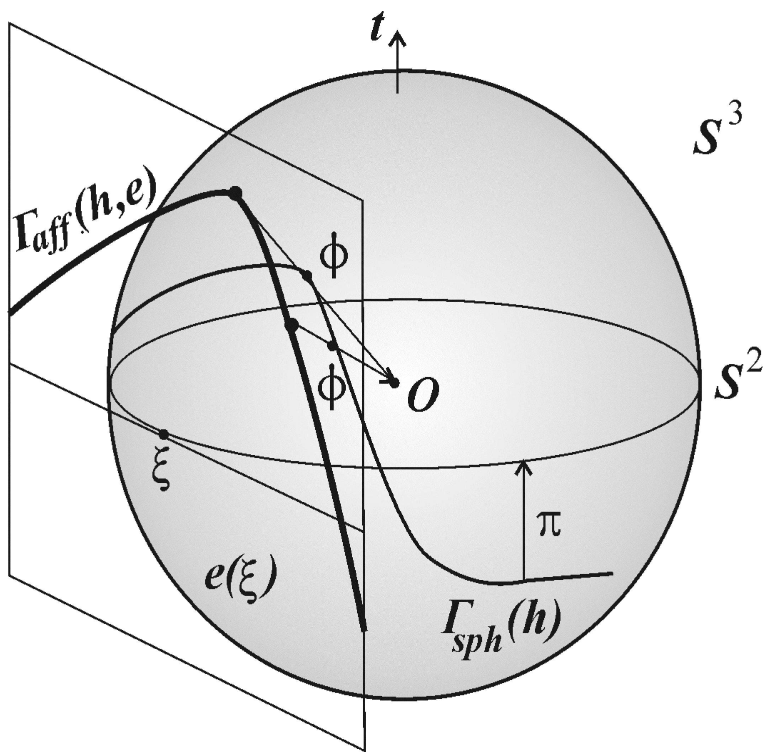

Figure 2

Collection of affine graphs of h**.** For each $\xi \in S^2$, denote by $e(\xi)$ the plane in $\mathbb{R}^3$ tangent to S^2 at the point ξ. Denote by $h|_e$ the restriction of h on the plane $e = e(\xi)$.

Consider the *affine graph* of the restriction $h|_e$, namely,

$$\Gamma_{aff}(h,e) := \{(v,u,t) \in \mathbb{R}^3 \mid (v,u) \in e; t = h(v,u)\}.$$

The union of all images of affine graphs $\Gamma_{aff}(h,e)$ on S^3 under the central projection ϕ with the center O equals the spherical graph of h (see Fig. 2).

Definition 2.2. A function h is called *hyperbolic* if $\Gamma_{aff}(h, e(\xi))$ is a saddle surface for every $\xi \in S^2$. A hérisson (or a virtual polytope) is called *hyperbolic* if its support function is hyperbolic.

The spherical graph of h and the collection of affine graphs of h have the same convexity properties. More precisely,

(1) all affine graphs of h are saddle surface if and only if the spherical graph of h is a spherically saddle surface;
(2) inflection arches (see Definition 3.1) of the spherical graph correspond to inflection rays of affine graphs.

Horns of hyperbolic objects.

Definition 2.3. Let H be a hyperbolic virtual polytope or a hyperbolic hérisson. A point $P \in H$ is called a *horn* if there exists a plane e passing through P and intersecting the surface H locally just at one point P.

Let e^+ be the half-space bounded by e and containing a neighborhood of P on the surface H. The outward normal vector n of e^+ is called an *outward vector* of the horn P.

Definition 2.4. Denote by $N(P)$ the set of all outward vectors of P. A vector d is called a *direction vector* of the horn P if $(d,n)>0$ for each $n \in N(P)$ (here and in the sequel $(\cdot,\cdot)$ stands for the scalar product).

The set of direction vectors of a horn is always non-empty.

Lemma 2.5. *Let H be a hyperbolic herisson or a hyperbolic polytope. Let $\{d_i\}$ be a collection of direction vectors of its horns (we take one direction vector for each horn). Then*

$$\bigcup_i S^+(d_i) = S^2,$$

where $S^+(d) = \{x \in S^2 : (x,d) > 0\}$. □

3. Inflection arches. Möbius-type theorems for two-dimensional saddle spheres in S^3

Definition 3.1. Let Γ be a smooth saddle surface in S^3.

An *inflection arch* of the surface Γ is a great semicircle $S \subset S^3$ such that

- $S \subset \Gamma$;
- for each great 2-dimensional sphere $e \subset S^3$ which intersects S transversely, the point $e \bigcap S$ is an inflection point of the curve $e \bigcap \Gamma$.

An inflection arch carries a natural orientation (see Fig. 3).

Theorem 3.2. *Let Γ be a two-dimensional closed smooth saddle surface lying in S^3 and admitting the bijective orthogonal projection onto some great sphere $S^2 \subset S^3$. Assume that Γ is non-degenerate, i.e., it does not coincide with a great sphere. Then*

1. *Γ contains at least 4 disjoint inflection arches.*
2. *The projections of all inflection arches onto S^2 form an arrangement of disjoint oriented great semicircles $\{A_i\}$ such that*

$$\bigcup_i S^+(A_i) = S^2,$$

   where $S^+(A_i)$ is the hemisphere bounded by the extension of A_i consistent with the orientation of A_i.
3. *The arrangement $\{A_i\}$ contains at least one subarrangement which equals (up to an isotopy and a symmetry) one of the arrangements $\mathfrak{A}_1$, $\mathfrak{A}_2$ presented in Fig. 8. By this reason, the arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are called the basic arrangements.*

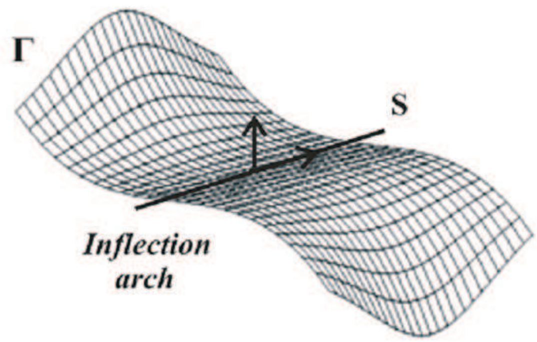

FIGURE 3

Proof. **(1)** By the assumption, Γ is the spherical graph of the support function h of some hyperbolic hérisson H. The map $\varphi : \mathbb{R}^3 \setminus O \to \mathbb{R}^3$, given by the formula

$$\varphi(\cdot) = (h'_x(\cdot), h'_y(\cdot), h'_z(\cdot))$$

maps $\mathbb{R}^3 \setminus O$ to the surface H. The surface H spans affinely the space $\mathbb{R}^3$ (see [11]). Therefore it has at least 4 horns, say, $P_1, ...,$ and P_4. Treat them separately.

For the horn P_1, fix a Cartesian coordinate system (x, y, z) in $\mathbb{R}^3$ such that $O = P_1$, and such that the x-coordinate of each point lying on H is positive. Therefore, $h'_x > 0$ everywhere except for the preimage of the horn P_1.

Let $\xi \in \varphi^{-1}(P_1)$. Choose the plane $E = E_\xi$ such that $\xi \in E$ and E contains a line parallel to the axes (x).

Denote by f the restriction of the function h to the plane E and denote by $F = F(\xi)$ the graph of the restriction. Let the Cartesian coordinate system (u, v, w) be such that $E = (u, v)$, $\xi = (0, 0)$ and the axes u is parallel to the axes x.

By construction, the following statements are valid:

(1) $f'_u(u, v) \geq 0$.
(2) $f'_u(u, v) = 0 \Leftrightarrow (u, v) \in \varphi^{-1}(P_1)$.
(3) $f'_u(0, 0) = 0$.
(4) $f(\xi) = f(0, 0) = 0$.
(5) The surface F is saddle.
(6) The surface F tangents E at the point ξ.

Denote by $T \subset E$ the set of points at which the surface F tangents the plane E. Obviously, $T = \varphi^{-1}(P_1) \cap E$.

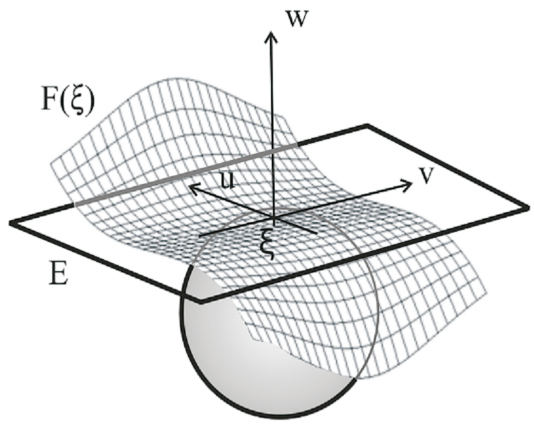

FIGURE 4

We show below that the above properties (1) and (5) of F imply that the set T is bigger than just one point ξ and contains some ray.

For a plane $E(\alpha)$ passing through the points $(0,0,0)$ and $(0,\varepsilon,f(\varepsilon))$, take the connected component $C(\alpha)$ of the set $F\setminus E(\alpha)$ which lies next to the point $(0,0)$ in the positive u-direction (i.e., the component which contains points with zero v-coordinate and small positive u-coordinates.)

For an appropriate choice of $E(\alpha)$, the component $C=C(\alpha)$ is infinite in both directions along the v-axes.

Indeed, since F is saddle, C can not bounded. For some choice of the plane $E(\alpha)$, the component C is infinite to the left along the axes v. Similarly, for some other choice, C is infinite to the right along the axes v. Therefore, for some intermediate choice of $E(\alpha)$, the component $C(\alpha)$ is infinite in both directions.

Denote by (α) the orthogonal projection of $C(\alpha)$ onto the plane E. Following A.V. Pogorelov, study its behaviour, as $\alpha\to 0$.

A point $(v_0,u_0)\in E$ belongs to T if and only if $(v_0,u_0)\in\Gamma\cap E$ and for all small $\alpha\geq 0$, the line in the plane E given by $v=v_0$ intersects the boundary of α at least twice.

At this point, A.V. Pogorelov erroneously concluded that the set (α) looks like in Case 1 (see Fig. 5). He deduced then that the tangent set T contains a line. Similar treatment of the other horns implied then that the surface Γ is a great sphere.

Pogorelov missed the Case 2 (see Fig. 5).

Lemma 3.3. *If the set T does not contain a line, then it is restricted by graphs of two functions, say, p_1 and p_2. These functions are defined on a ray, say on the ray $[a,\infty)$. Besides, $p_1(a)=p_2(a)$, and the function p_1 (respectively, p_2) is concave down (respectively, concave up).*

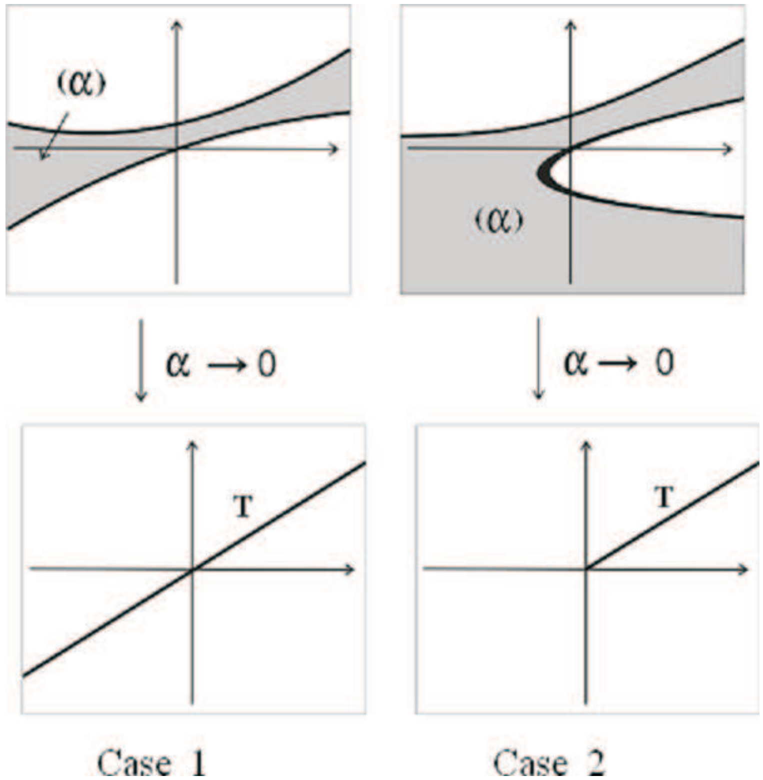

FIGURE 5

Proof. Suppose the contrary. Then there exists a line l in the plane E which cuts a bounded connected component off the set T which does not contain the endpoint of T. Rotate the plane E around l on an angle δ such that the upper half-plane raises. For a small δ, the plane obtained cuts a bounded connected component off the surface Γ, which is impossible for a saddle surface. The lemma is proven.

Lemma 3.3 implies that the intersection of $\varphi^{-1}(P)$ with the plane $E(\xi)$ contains a ray. Treating similarly other points ξ (and therefore, other planes $E(\xi)$), we conclude that the surface Γ contains a great semicircle which corresponds to the horn P_1.

The other horns give at least three more semicircles lying on Γ. They are disjoint because they are contained in preimages of different points.

(2) This statement follows directly from Lemma 2.5.

(3) Put $S_i^- = S^2 \setminus S_i^+$. The statement (2) is equivalent to the identity $\bigcap_i S_i^- = \emptyset$. Show that the same identity is valid for some 4 hemispheres from the arrangement. Assume the contrary, i.e., that each 4 hemispheres have a common point. Then by Helly's Theorem, all hemispheres have a non-empty intersection. A contradiction.

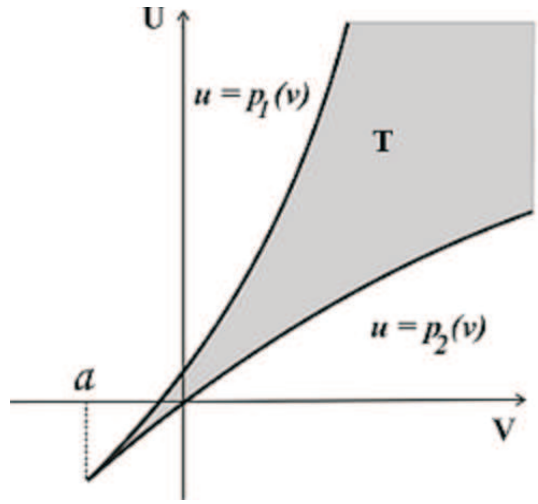

FIGURE 6

It remains to observe that there exist just two types (up to an isotopy and a symmetry) of arrangements of four oriented great semicircles satisfying $\bigcup_{i=1}^{4} S^+(d_i) = S^2$. $\square$

Corollary 3.4. *Each hyperbolic hérisson H generates an arrangement of disjoint oriented great semicircles on the unite sphere S^2. There is a natural one-to-one correspondence*

"semicircles of the arrangement $\leftrightarrow$ horns of the herisson $\leftrightarrow$ inflection arches of the graph of the support function h_H". $\square$

A *spherical polygon* in S^3 is a subset of some great sphere $S^2 \subset S^3$ bounded by a closed simple piecewise geodesic line.

Theorem 3.5. *Let Γ be a non-degenerate two-dimensional closed polytopal saddle surface lying in S^3 (i.e., all the facets of Γ are some spherical polygons). Suppose that Γ admits the bijective orthogonal projection onto some great sphere $S^2 \subset S^3$. Then Γ contains at least 4 disjoint facets $s_1, s_2, ..., s_k$ such that*

(1) *each of s_i is bounded by two convex broken lines (say, by L_1 and L_2) such that the convexity directions look like in Fig.7;*
(2) *each s_i contains a great semicircle;*
(3) *the surface Γ is concave up along one of the broken lines L_1 and L_2. It is concave down along the other broken line.* $\square$

Proof. By the assumption, Γ is the spherical graph of the support function h of some virtual polytope H.

The surface H has at least 4 horns, say, $P_1, ...,$ and P_4. Show that the tiles of the fan Σ_h dual to the horns give the required collection of facets.

For the horn P_1, fix a Cartesian coordinate system (x, y, z) in $\mathbb{R}^3$ such that $O = P_1$, and such that the x-coordinate of each point lying on H is positive.

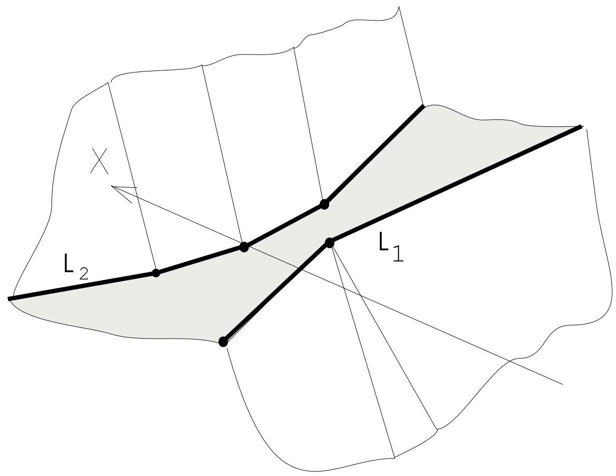

FIGURE 7

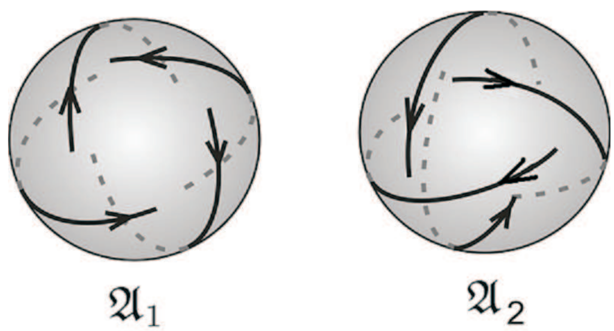

FIGURE 8

Therefore, $h'_x = 0$ on the tile of Σ_h which corresponds to the horn P_1, and $h'_x > 0$ elsewhere.

Let $\xi \in \varphi^{-1}(P_1)$. Choose the plane $E = E_\xi$ such that $\xi \in E$ and E contains a line parallel to the x-axes .

By construction, the graph F of the restriction $h\mid_E$ is horizontal above the tile which is dual to the horn P and has a positive slope in the direction of the x-axes. Besides, the surface F is saddle. These two properties imply the statement of the theorem. □

4. Two non-isotopic hyperbolic polytopes with 4 horns

Lemma 4.1. *The arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of oriented great semicircles presented in Fig. 8 are non-isotopic.*

Proof. Given an arrangement of great semicircles, construct a graph which is invariant under isotopies and symmetries. The vertices of the graph correspond to the great semicircles. Two vertices i and j are connected by an edge if either the extension of the great semicircle i in some direction first meets the

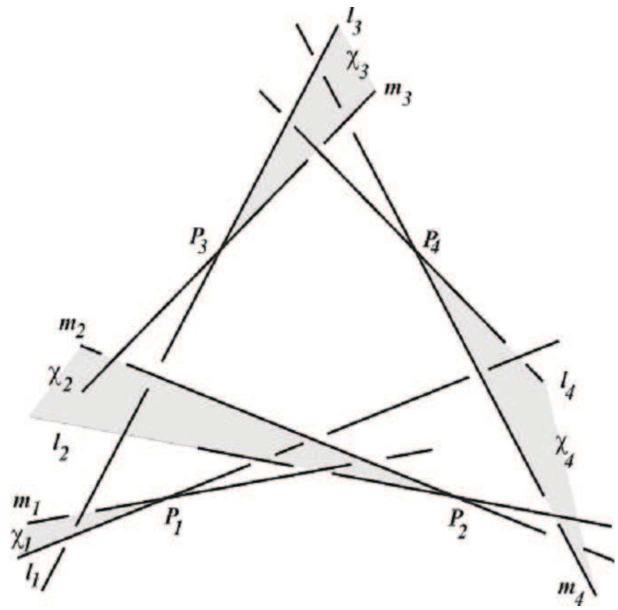

FIGURE 9

semicircle j, or the extension of the great semicircle j first meets the semicircle i. It remains to observe that the graphs for the arrangements in question are different. Namely, for the second arrangement we get a complete graph with four vertices, whereas for the first arrangement the graph is not complete. □

Example 4.2. (1) *There exist two hyperbolic polytopes, each with 4 horns, such that the generated arrangements are isotopic to $\mathfrak{A}_1$ and $\mathfrak{A}_2$.*
(2) *There exist two non-isotopic hyperbolic hérissons, which are smooth saddle (except for the four horns) saddle surfaces .*

Proof. The first hyperbolic polytope and the first smooth hyperbolic hérisson (see Fig. 1) are alreday presented by Martinez-Maure (see [7] and [6]).

It remains to construct the second hyperbolic polytope (steps 1-3) and then smoothen it (step 4) to obtain the second hyperbolic hérisson.

Step 1. Fix the positive and negative hemispheres $S^2_\pm$ with the poles $P = (0, 0, 1)$ and $-P = (0, 0, -1)$. Consider eight geodesic lines (i.e., great circles) in S^3 forming a linkage as is shown in Fig. 9.

This means that each pair of lines l_i, m_i has two common points P_i and $-P_i$. No other pairs of lines has intersections. Fig. 9 depicts the planar diagram of the linkage (i.e., its images under the projection π on the positive and negative hemispheres with indicated "passes"). In particular, the line l_1 passes over l_2 above S^2_+, the line l_1 passes under m_2 above S^2_+ ("under" and "over" refer to the direction of the t-axes). Denote by χ_i the spherical 2-gon with edges lying on l_i and m_i, assuming that its image is marked grey in Fig. 9. Each of these 2-gons has two vertices, namely, P_i and $-P_i$. The 2-gons $\Lambda_i = \pi(\chi_i)$ form a disconnected polytopal complex Λ embedded in S^2.

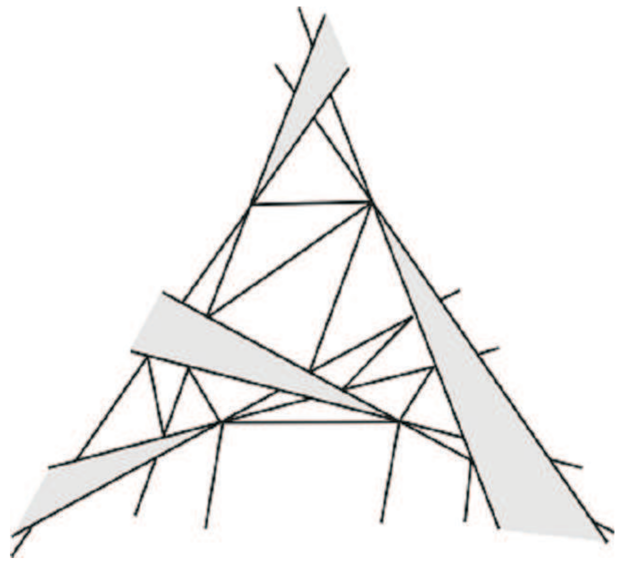

FIGURE 10

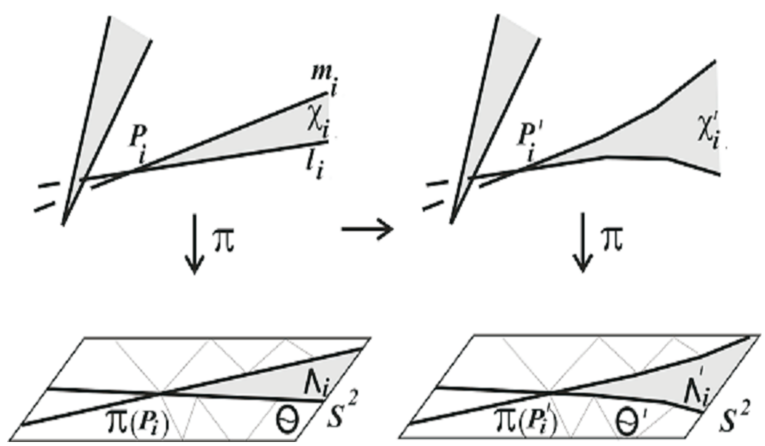

FIGURE 11

Fix a tiling Θ of S^2 as is shown in Fig. 10. It has four 2-gons (marked grey), whereas all other polygons are spherical triangles.

Step 2. Next, move somewhat the vertices of the tiling Θ (and denote the new points by the same letters with primes) in such a way that each of Λ_i is replaced by a spherical polytope Λ' bounded by two convex broken lines (see Fig. 11). The lines should be broken at each vertex of the tiling.

Apply the synchronized changes to χ_i. Namely, let χ_i' be the 2-dimensional spherical polygons lying close to χ_i such that $\pi(\chi_i') = \Lambda_i'$. In addition, we claim that the prolongations of the edges of χ_i' adjacent to P_i' (and $(-P_i)'$) and the boundary (broken) lines of χ_i' form the same linkage type as the original lines l_i, m_i.

The spherical polygons χ_i' play the role of inflection arches.

Step 3. There exists a unique piecewise linear function h such that

1. The function h is linear on each triangle of Θ' and on each Λ'_i (more precisely, h is linear on each cone in $\mathbb{R}^3$ based on these spherical polygons).

2. The polytopes χ'_i lie on its spherical graph $\Gamma_{sph}(h)$.

Show that the surface $\Gamma_{sph}(h)$ is saddle at each of its vertices A. If A does not equal any of the points P'_i or $(-P_i)'$, then A is a vertex of χ'_i for some i, and the angle of χ'_i at the vertex A is greater than π. This means that A is a saddle point.

Assume that $A = P'_1$ can be cut off. By construction, the surface in question contains four segments with an endpoint at P'_1: the two adjacent edges of λ'_1 and the segments corresponding to their extensions. Due to the linking type, each hemisphere, whose boundary passes through P'_1, contains at least one of the segments. Therefore, P'_1 is a saddle vertex as well. The other vertices $(-P_i)'$ are treated similarly. The statement (1) is proven.

Step 4. The previous steps yield a saddle surface which can be interpreted as the graph of the support function of some virtual polytope χ'_i. It is a closed polytopal surface in $\mathbb{R}^3$ which is saddle at each of its vertices except for 4 horns.

All the vertices of K, except for the horns, have the valence 3. The smoothing technique developed in [8] and [10] allows to construct a smooth saddle surface with 4 horns approximating the surface K. $\square$

R E F E R E N C E S

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