# POINTED SPHERICAL TILINGS AND HYPERBOLIC VIRTUAL POLYTOPES

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ABSTRACT. The paper presents an introduction to the theory of hyperbolic virtual polytopes from the combinatorial rigidity viewpoint. Namely, we give a shortcut for a reader acquainted with the notions of Laman graph, 3D lifting, and pointed tiling.

From this viewpoint, a hyperbolic virtual polytope is a stressed pointed graph embedded in the sphere $S^2$.

The advantage of such a presentation is that it gives an alternative and the most convincing proof of existence of hyperbolic polytopes.

## 1. Introduction

In this paper, we give an alternative presentation of the theory of hyperbolic virtual polytopes.

The reader should not confuse them with polytopes lying in a hyperbolic space. In the context of the paper, the term "hyperbolic" means "saddle". In some sense, hyperbolic polytopes are opposite to convex polytopes by their convexity property.

This theory arose originally as a tool for constructing counterexamples (see [4, 11, 12, 20]; see also the very first counterexample constructed without hyperbolic polytopes [9]) to the following uniqueness conjecture, proven by A.D. Alexandrov (see [1]) for analytic surfaces.

**Uniqueness conjecture for smooth convex surfaces.**
*Let $K \subset \mathbb{R}^3$ be a smooth closed convex surface. If for a constant $C$, at every point of $\partial K$, we have $R_1 \leq C \leq R_2$, then $K$ is a ball. ($R_1$ and $R_2$ stand for the principal curvature radii of $\partial K$).*

We refer the reader to [11] for the relationship between the conjecture and the theory of hyperbolic polytopes.

By a *convex polytope* we mean the convex hull of some finite set of points. Denote by $\mathcal{P}$ the set of all convex polytopes in $\mathbb{R}^3$. Equipped with the Minkowski addition $\otimes$, the set $\mathcal{P}$ is a commutative semigroup with the unit element $\{O\}$. The set of all formal Minkowski differences $\mathcal{P}^* = \{K \otimes L^{-1} \mid K, L \in \mathcal{P}\}$ is a group which is called the *group of virtual polytopes*.

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Similarly to rational fractions, we identify $K \otimes L^{-1}$ and $K' \otimes (L')^{-1}$ whereas $K \otimes L' = K' \otimes L$.

The elements of $\mathcal{P}^*$, which are called *virtual polytopes*, are not mere formal expressions. They can be interpreted geometrically, and multiple geometric interpretations are crucial for their study.

The first geometric interpretation appeared in the paper [6]. From its viewpoint, a virtual polytope is a piecewise constant function with some specific properties (a *convex chain*).

Alternatively, in the framework of the present paper, a virtual polytope is a stressed spherically embedded graph. We turn the set of all stressed graphs into a group (Section 3), which is shown (Theorem 3.7) to be canonically isomorphic to the group of virtual polytopes.

Further, among the virtual polytopes we single out the class of *hyperbolic virtual polytopes* (for short, hyperbolic polytopes).

Very roughly, hyperbolic polytopes are defined to be as non-convex as possible. By definition, the graph of the support function of a hyperbolic polytope is a saddle surface (in contrast to convex polytopes, for which the graph of the support function is a convex surface).

The crucial link to the pointed tilings is the following: if a spherically embedded stressed graph is pointed, then the corresponding virtual polytope is hyperbolic (Lemma 4.3).

The theory of hyperbolic polytopes has the following curious feature: the most non-trivial and important fact is the existence and diversity of hyperbolic polytopes (see [20] for some 3D images). In other words, it took a lot of efforts to construct different examples of hyperbolic polytopes.

The advantage of the approach of the paper is that it gives an alternative and the most convincing proof of existence of hyperbolic polytopes.

The paper first pulls the theory of planar pointed tilings to the sphere $S^2$. Necessary facts of graphs rigidity are transferred onto the sphere due to some simple adjustments of Section 2 and the papers [2] and [18]. The only difference between the spherical and the planar case (which however changes the situation very much) is the existence of *pseudo-di-gons*. Namely, each planar polygon has at least three convex angles, whereas on the sphere there exist polygons with just two convex angles (see Fig. 4).

This fact changes Laman-type counts for pointed tilings. As a consequence, there exist pointed spherically embedded Laman-plus-one (and even Laman-plus-$k$ graphs (see Example 4.5 and Example 4.6). They possess a non-trivial saddle 3D lifting. By definition, this is nothing but a hyperbolic polytope.

Thus a hard problem of constructing hyperbolic polytopes (which originally were 3D objects) is reduced to construction of a spherically embedded pointed graph.

This technique has already led to a new result. Namely, the author obtained a refinement of A.D Alexandrov theorem on 3D polytopes with mutually non-insertable faces (see [14]).

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## 2. Graphs on the sphere. Space of equilibrium stresses.

A *graph* is a pair $G = (V, E)$, where $V = \{1, 2, ..., n\}$ is a finite set, $E$ is a set of unordered pairs $(i, j)$ such that $i, j = 1, ..., n$, and $i \neq j$. The elements of $V$ and $E$ are called *vertices* and *edges* respectively.

A subgraph $G'$ of $G$ is called *proper* if $G \neq G'$.

By a *graph embedded in* $R^3$ we mean a triple $G = (V, E, p)$, where $V$ and $E$ are as above, and $p$ is an injective mapping $p : V \to \mathbb{R}^3$.

The points $p(i)$ are denoted for short by $p_i$ and are called *vertices* of the graph. The segments $p_ip_j$ for $(i, j) \in E$ are called the *edges* of the graph and are assumed to be non-crossing.

Denote by $S^2 \subset \mathbb{R}^3$ the unit sphere centered at $O$. Its points we identify with their radius vectors.

By a *spherically embedded graph* we mean a quadruple $G = (V, E, p, l)$, where $V$ and $E$ are as above, $p$ is an injective mapping $p : V \to S^2$. The points $p_i = p(i)$ are called *vertices* of the graph. A bit more care is needed here to define edges.

The function $l$ defined on the set $E$ maps each pair $(i, j) \in E$ to some geodesic segment with endpoints at $p_i$ and $p_j$. The segments $l(i, j)$ are denoted for short by $l_{ij}$ and are called *edges* of the graph. We don't claim that $l_{ij}$ are the shortest geodesic segments (i.e., the minor arc of a great circle) connecting $p_i$ and $p_j$, so there are two possible edges with fixed endpoints (or even infinitely many possible edges for antipodal endpoints).

We assume that the edges $l_{ij}$ are non-crossing.

Besides, in the section, we assume that all embeddings are *generic* [3]. In its general stating this means that the vertex coordinates are algebraically independent. In particular, this means that for a spherically embedded graph, there are no antipodal vertices.

**Example 2.1.** *It is convenient to consider a great circle on $S^2$ as an embedded graph (with no vertices and a single closed edge) as well. We call it an exotic graph EG.*

We will use a slightly modified (in comparison with [3]) definition of an equilibrium stress of a graph $G$ in $R^3$ and its natural adjustment for a spherically embedded graph. However, the below definition is in some sense equivalent to the classical one.

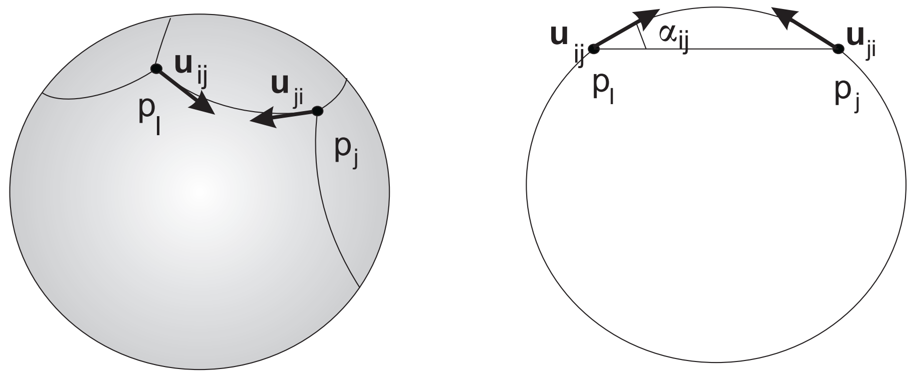


FIGURE 1

**Definition 2.2.** Let $G = (V, E, p)$ be a graph embedded in $\mathbb{R}^3$. A mapping $s : E \to R$ is called an *equilibrium stress* (or shortly, a *stress*) of $G$ if for each $i$, we have

$\sum_{(ij)\in E} s(i, j)\mathbf{u}_{ij} = 0$, where $\mathbf{u}_{ij} = \frac{\overrightarrow{p_i p_j}}{|p_i p_j|}$.

A stress is called *non-trivial* if it is not identically zero.

A stress is called *non-zero* if it is non-zero on each edge.

The space of all stresses of $G$ we denote by $\mathcal{S}(G)$.

**Definition 2.3.** Let $G = (V, E, p, l)$ be a spherically embedded graph. A mapping $s : E \to R$ is called an *equilibrium stress* (or shortly, a *stress*) of $G$ if for each $i$, we have

$\sum_{(ij)\in E} s(i, j)\mathbf{u}_{ij} = 0$, where $\mathbf{u}_{ij}$ are the unit vectors tangent to $l_{i,j}$ at the point $p_i$. Their direction are chosen as is depicted in Fig. 1.

The space of all stresses of $G$ we denote by $\mathcal{S}(G)$.

**Definition 2.4.** We assume that the exotic graph $EG$ possesses a stress. It can be any real number assigned to its only edge.

The following construction reduces the stress of a spherically embedded graph $G$ to a stress of some graph embedded in $R^3$. The ideas are borrowed from [2] and [18].

Given a graph $G$ embedded in $S^2$, we add the point $p_{n+1} = O$ as a new vertex. We next replace the edges of $G$ by corresponding line segments. Finally, we add the edges $(i, n + 1)$ for $i = 1, ..., n$ as new edges and denote the embedded graph obtained by $\overline{G} = (\overline{V}, \overline{E}, \overline{p})$.

**Proposition 2.5.** *The spaces of stresses of $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$ are canonically isomorphic.*

Proof. Let $s$ be a stress of $G$. Define the stress $\overline{s}$ of $\overline{G}$ as follows. For $i, j < n + 1$, let $\alpha_{i,j}$ be the angle between $\overrightarrow{p_i p_j}$ and $\mathbf{u}_{ij}$ (see Fig. 1).

Put $\overline{s}(i, j) = \begin{cases} s(i, j)/\cos\alpha_{ij} & \text{if} \quad |l_{i,j}| \le \pi, \\ -s(i, j)/\cos\alpha_{ij} & \text{otherwise.} \end{cases}$

Put also $\overline{s}(i, n+1) = -\sum_{j=1}^{n} s(i,j) \tan \alpha_{ij}$. Show that his mapping is an isomorphism between $\mathcal{S}(G)$ and $\mathcal{S}(\overline{G})$. We check first that $\overline{s}$ is a stress of $\overline{G}$. The condition $\sum_{(ij)\in E} s(i,j)\mathbf{u}_{ij} = 0$ at a vertex $p_i$ for $i \le n$ is valid by construction.

Besides, the sum of all vectors $\overline{s}(i,j)\mathbf{u}_{ij}$ equals zero. Therefore, the condition $\sum_{(ij)\in E} s(i,j)\mathbf{u}_{ij} = 0$ is also valid for the vertex $p_{n+1} = O$. To conclude the proof, observe that the described above mapping $\mathcal{S}(G) \to \mathcal{S}(\overline{G})$ is invertible. That is, given a stress $\overline{s}$ of $\overline{G}$, the stress $s$ can be restored. □

**Definition 2.6.** ( [3]) A graph $G = (V, E)$ with $n$ vertices and $m$ edges is a Laman graph if

- $m = 2n - 3$, and
- each subset $V'$ of $k$ vertices spans not more than $2k - 3$ edges. (We say that an edge $(i,j) \in E$ is *spanned* by $V'$ if $i, j \in V'$.)

**Definition 2.7.** ( [7]) A Laman graph with one extra edge is a *Laman-plus-one graph*. Similarly, a Laman graph with $k$ extra edges is a *Laman-plus-k graph*.

**Definition 2.8.** ( [7]) A graph $G$ is a *rigidity circuit* if the removal of any of its edges yields a Laman graph. Equivalently, $G$ is a rigidity circuit if it is a Laman-plus-one graph and has no Laman-plus-one proper subgraph.

The below proposition is a spherical version of some classical facts.

**Proposition 2.9.** *Let $G$ be a generic spherically embedded graph.*

(1) *If $G$ is a Laman graph, then $G$ is infinitesimally rigid.*
(2) *If $G$ is a Laman-plus-one graph, then $G$ possesses a non-trivial (i.e., not identically zero) stress.*
(3) *If $G$ is a rigidity circuit, then $G$ possesses a non-zero stress.*

Proof. 1. The rigidity of generic Laman graphs is valid for graphs embedded in the plane (see [3]). The paper [18] proves that it is also valid for spherically embedded graphs. More precisely, it is proven that infinitesimal motions of a spherically embedded graph are in a one-to-one correspondence with the infinitesimal motions of its projection on the plane.

The paper [18] treats only those spherical embeddings that fit on an open hemisphere. Still the general case is easily reduced to a hemispherical one via the following trick.

Fix a hemisphere $S^+$. For a spherically embedded graph $G = (V, E, p, l)$, construct the new graph $G^+ = (V, E, p^+, l^+)$ such that $p_i^+ \in S^+$, and $p_i^+ = \pm p_i$ depending on which of the points $p_i$ and $-p_i$ belongs to $S^+$. Finally, $l_{ij}$ is defined to be the segment lying also in $S^+$.

This mapping preserves rigidity but does not maintain non-crossing property.

2. Denote by $n$ the number of vertices of $G$ and by $m$ the number of its edges. In [18] it is proven that $G$ is infinitesimally rigid. Together with Corollary 2.3.1 from [3] applied to the graph $\overline{G}$, this directly implies that

$6 = 3(n+1) - (m+n) + dim(\mathcal{S}(\overline{G}))$.

Therefore, $dim(\mathcal{S}(\overline{G})) = 1$.

3. Suppose the contrary, i.e., that $G$ has a non-trivial stress which admits zero values on some of the edges. Removal of the zero stressed edges yields a proper subgraph of $G$ with a non-trivial stress. It is at least a Laman-plus-one graph. A contradiction. □

## 3. 3D liftings for graphs on the sphere

A (spherical) polygon on the sphere $S^2 \subset \mathbb{R}^3$ is a domain of $S^2$ bounded by a closed non-crossing polygonal line (its edges are assumed to be geodesic arcs).

A spherical polygon $A$ spans a cone $C(A)$ in $\mathbb{R}^3$ with the apex at $O$. Namely, we put $C(A) = \{\lambda x \in \mathbb{R}^3 \mid \lambda \in \mathbb{R}^+,\ x \in A\}$.

A spherically embedded graph $G$ generates a tiling $\mathcal{ST}(G)$ of $S^2$. Each tile gives a cone, and thus $\mathcal{ST}(G)$ yields a tiling of $\mathbb{R}^3$ into the union of cones:

$\mathcal{CT}(G) = \{C(A) \mid A \in \mathcal{ST}(G)\}$.

**Definition 3.1.** A function $h : \mathbb{R}^3 \to \mathbb{R}$ is called a 3D *lifting* of a spherically embedded graph $G$ if it possesses the four properties:

(1) $h$ is continuous;
(2) $h(O) = 0$;
(3) $h$ is piecewise linear;
(4) $h$ is linear on each cone of the tiling $\mathcal{CT}(G)$.

A 3D lifting is *non-trivial* if it is not a globally linear function.

A 3D lifting is *tight* if it is not a 3D lifting of some proper subgraph of $G$. That is, a tight lifting is not linear in neighborhood of inner points of the edges of $G$.

Given a graph $G$, the set of all its 3D liftings form a linear space.

**An important example.** We will show that a convex polytope yields canonically a positively stressed spherically embedded graph.

Let $K \subset \mathbb{R}^3$ be a convex polytope. Remember that its support function $h_K : \mathbb{R}^3 \to \mathbb{R}$ is defined by $h_K(\mathbf{x}) = max_{y \in K}(\mathbf{x}, \mathbf{y})$, where $(\mathbf{x}, \mathbf{y})$ stands for the scalar product. The support function is known to satisfy the four properties from Definition 3.1 with respect to some conical tiling of $\mathbb{R}^3$ (the *outer normal fan* of $K$. Being intersected with $S^2$, the conical tiling generates a tiling $\Sigma_K$ of the sphere $S^2$, which we call the *spherical fan* of $K$. Its 1-skeleton is some spherically embedded graph $G_K$.

The polytope $K$ and its fan $\Sigma_K$ are combinatorially dual. In particular, the edges of $G_K$ are in one-to-one correspondence with the edges of $K$.

**Proposition 3.2.** *Let $K \subset \mathbb{R}^3$ be a convex polytope. In the above notation, we have:*

(1) *The support function $h_K$ is a tight 3D lifting of the graph $G_K$.*

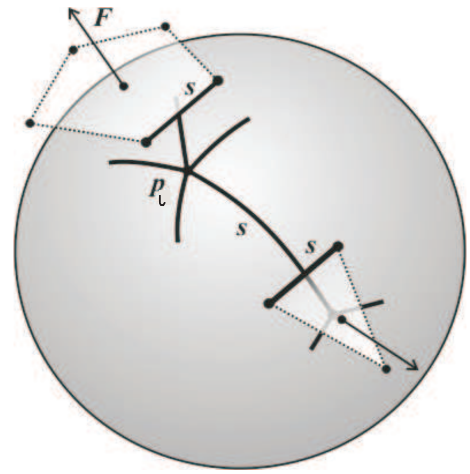


FIGURE 2

(2) *For each plane $e \subset \mathbb{R}^3$, the restriction $h_K|_e$ of the support function $h_K$ on the plane $e$ is a convex function. Equivalently, the graph of $h_K|_e$ is concave down.*
(3) *The function $s_K$ which maps each edge of the graph $G_K$ to the length of the corresponding edge of $K$ is a positive stress of $G_K$.*
(4) *Vice versa, given a spherically embedded graph $G$ with a positive stress $s$, there exists a unique (up to a translation) convex polytope $K \subset \mathbb{R}^3$ such that $G = G_K$ and $s = s_K$.*

Proof. The proposition is a mere reformulation of some classical facts on convex polytopes for which we refer the reader to [19] and [2] for advanced details. (1) reformulates the definitions of the outer normal fan and support function. (2) means just the convexity of $h_K$.

The statement (3) is obvious. Indeed, let $p_i$ be a vertex of $G_K$. By duality, it corresponds to a face $F$ of $K$ such that the outer normal of $F$ equals $p_i$. The edges of $F$ correspond by duality (and are orthogonal) to the edges of $G_K$ incident to the vertex $p_i$ (see Fig. 2). The condition of the Definition 2.3 means that the sum of edge vectors of the polygon $F$ equals zero, which is true.

Prove (4). By the above reason, a positively stressed graph $G$ yields a collection of convex polygons (for each vertex $p_i$, we have a polygon) which can be patched together to form a convex polytope (see Fig. 2). □

**Example 3.3.** *In the framework of Proposition 3.2 (4), a positively stressed exotic graph EG generates a line segment. Its length equals the value of the stress.*

Denote by $\mathcal{SG}$ the set of all pairs of type
(a spherically embedded graph $G$; a non-zero stress $s$ of the graph $G$).
To avoid degenerate cases, we claim that each vertex of $G$ is at least trivalent.

Exotic graphs and the empty graph are also included. The following definition turns $\mathcal{SG}$ to a group which is called the *group of stressed graphs.*

**Definition 3.4.** The sum of two stressed graphs $(G; s) = (G_1; s_1) + (G_2; s_2)$ is defined via the following procedure:

- Taken together, the tilings $\mathcal{ST}(G_1)$ and $\mathcal{ST}(G_2)$ generate their common refinement, some new tiling of $S^2$. There appear new vertices, and some of the edges get split. The 1-skeleton of the common refinement can be viewed as a spherically embedded graph $G$.
- $G$ has a natural stress defined as the sum of $s_1$ and $s_2$. More precisely, let $l$ be an edge of $G$. If it lies on some edge of $G_1$ and on no edge of $G_2$, then we assign to $l$ the stress inherited from $s_1$. If it lies on an edge of $G_1$ and on an edge of $G_2$, we take the sum of inherited stresses. However, the stress is not necessarily non-zero, so we need some further reductions.
- To make the stress non-zero, we remove all zero stressed edges of $G$. On this previous step, sometimes appear *redundant vertices* of two types. The vertices of the first type are those possessing just two adjacent edges. In this case the edges form the angle $\pi$ and are equally stressed.

  The redundant vertices of the second type are isolated vertices.
- We remove all redundant vertices.
- The stressed graph obtained is called the sum of the stressed graphs $(G_1; s_1)$ and $(G_2; s_2)$.

**Remark 3.5.** Exotic graphs and the empty graph fit nicely in this scheme. An exotic graph can be represented as a sum of two non-exotic ones. This means that without them we would fail to get a group.

**Proposition 3.6.** *Each stressed graph $(G; s) \in \mathcal{SG}$ is the difference of some two positively stressed graphs from $\mathcal{SG}$.*

Proof. For each edge $l_{i,j}$ of $(G; s)$ with a negative stress $s$, we add to $(G; s)$ a positively stressed exotic graph whose edge contains $l_{i,j}$. (the stress should be greater or equal than $-s$). This makes the sum positively stressed. $\square$

Summarizing the above, we get the following theorem.

**Theorem 3.7.** (1) *The group of stressed graphs $\mathcal{SG}$ is generated by $\{(G_K; s_K)\}$, where $K$ ranges over the set of convex polytopes in $\mathbb{R}^3$.*

(2) *The group of stressed graphs $\mathcal{SG}$ is canonically isomorphic to the group of virtual polytopes $\mathcal{P}$ (see Section 1).*

(3) *Therefore, we arrive at the same group of virtual polytopes as it was defined by A. Pukhlikov and A. Khovanskii in* [6]. $\square$

**Definition 3.8.** Keeping in mind the canonical isomorphism from Theorem 3.7, we will call an element of the group of stressed graphs *a virtual polytope represented by a stressed graph.*

**Theorem 3.9.** (1) *Given a spherically embedded graph $G$, the space of its stresses is canonically isomorphic to the space of its 3D liftings.*
(2) *A generic spherically embedded Laman-plus-k graph has a non-trivial 3D lifting for any $k = 1, 2, ...$*
(3) *A spherically embedded rigidity circuit has a tight 3D lifting.*

Proof. For the graph generated by a convex polytope, (1) follows from Proposition 3.2 and Proposition 3.6. The general statement follows by linearity and Proposition 3.6.

(2) and (3) follow from Theorem 3.7 and Proposition 2.9. □

This theorem motivates the the following definition.

**Definition 3.10.** In the framework of Theorem 3.9, the 3D lifting $h = h(G; s)$ of a virtual polytope represented by a stressed graph $(G; s)$ is called the *support function* of $(G; s)$.

This definition is consistent with the definition of the support function of a convex polytope $K$; that is, $h_K = h(G_K, s_K)$.

## 4. Pointed graphs and hyperbolic virtual polytopes

Now we are ready to single out the class of *hyperbolic virtual polytopes*.

**Definition 4.1.** A surface $F \subset \mathbb{R}^3$ is called *a saddle surface* if there is no plane cutting a bounded connected component off $F$.

Equivalently, a surface $F$ is *saddle* if no plane intersects $F$ locally at just one point.

**Definition 4.2.** A function $h : \mathbb{R}^3 \to \mathbb{R}$ is called *hyperbolic* if the graph of its restriction $h|e$ to any plane $e$ is a saddle surface.

A virtual polytope represented by a stressed graph $(G; s)$ is called *hyperbolic* if the induced 3D lifting $h$ is hyperbolic.

A spherically embedded graph is called *pointed* if each of its vertices is incident to an angle larger than $\pi$ (see Fig. 3).

Hyperbolic polytopes and pointed graphs are closely related due to the following simple fact.

**Lemma 4.3.** [11] *Given $(G; s) \in \mathcal{SG}$, if $G$ is pointed, then $(G; s)$ is hyperbolic.* □

We borrow the below definitions and proposition (including the idea of its proof) from the theory of planar pointed pseudo-triangulations (see [16, 17]).

A spherical polygon is called a *pseudo-triangle* (respectively, *pseudo-di-gon*) if it has exactly three (respectively, exactly two) angles smaller than $\pi$.

**Proposition 4.4.** *Let $G$ be a spherically embedded graph with $n$ vertices and $m$ edges. Suppose that each tile of $\mathcal{T}(G)$ is either a pseudo-triangle or a pseudo-di-gon. Then $m = 2n - 6 + d$, where $d$ is the number of pseudo-di-gons in the tiling $\mathcal{T}(G)$.*

FIGURE 3. A pointed graph

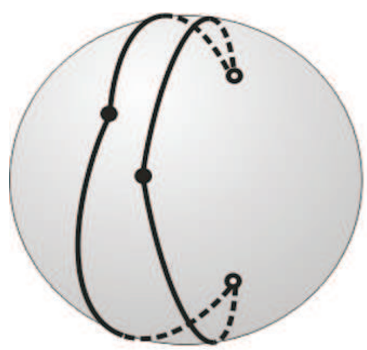

FIGURE 4. A pseudo-di-gon

Proof. Denote by $c$ the total number of convex angles (i.e., the angles smaller than $\pi$) of all tiles from $\mathcal{T}(G)$. Denote by $t$ the number of pseudo-triangles. We have

$n - m + d + t = 2$ (Euler's formula),

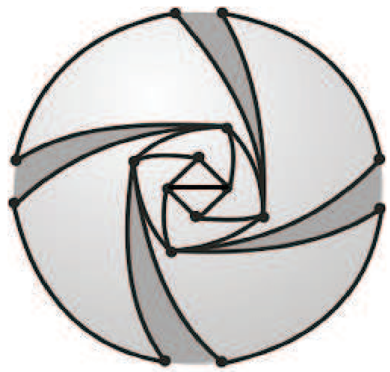

FIGURE 5. A pointed rigidity circuit. The figure depicts one side of the sphere, the other side looks analogously.

$c = 2d + 3t$ (first count of convex angles), and
$c = 2m - n$ (second count of convex angles),
which imply together the required. □

Since we aim at hyperbolic polytopes, we are interested in stressed pointed embedded graphs.

Recall that a planar pointed graph never has a non-zero stress (see [17]). We sketch here the proof which appeals to the theory of saddle surface.

If a pointed graph has an equilibrium stress, then it has a 3D lifting. Hence its graph is a piecewise linear surface which is saddle (due to the pointed property) and which coincides with the plane everywhere except for a bounded set. The latter is impossible.

The crucial property of pointed spherically embedded graphs is that some of them (actually, many of them) have a non-trivial 3D lifting. This means that there exist many hyperbolic virtual polytopes.

**Example 4.5.** *Figure 6 presents a spherically embedded rigidity circuit. It has 24 vertices and 46 edges. The graph generates a tiling with four pseudo-di-gons (marked grey). Due to Proposition 2.9, it has a tight 3D lifting.*

In the framework of the above theory, it becomes quite easy to construct a pointed rigidity circuit $G$. Indeed, we know in advance that the tiling $\mathcal{T}(G)$ should contain four pseudo-di-gons. So one has to place on the sphere four disjoint pseudo-di-gons and after that complete the drawing by a pointed pseudo-triangulation of their complement. This is not tricky at all. It should be

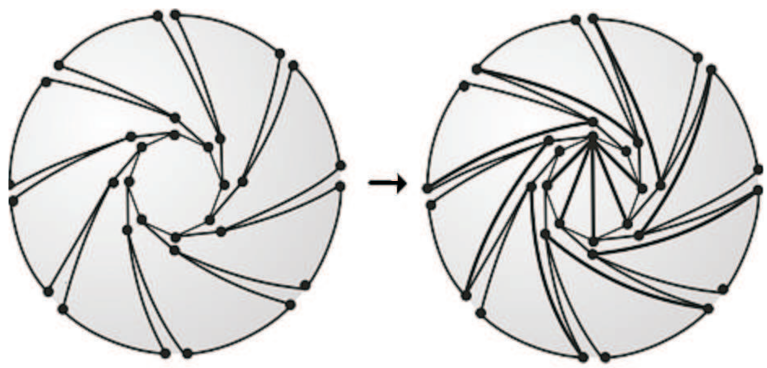

FIGURE 6. A pointed Laman-plus-5 graph

mentioned how much efforts were involved to construct the first examples of hyperbolic polytopes (see [11, 12, 9]).

**Example 4.6.** *Fig. 6 presents a procedure which leads to a pointed embedded Laman-plus-k graph (on the left). Its space of stresses is k-dimensional.*

**Example 4.7.** *Figure 7 presents another spherically embedded rigidity circuit.*

Similarly to the Example 4.5, the graph generates a tiling which has four pseudo-di-gons, but this time the pseudo-di-gons lie in a different position in the following sense.

It is easy to observe that each pseudo-di-gon contains a great semicircle. Given a pointed embedding of a rigidity circuit $G$, fix a great semicircle for

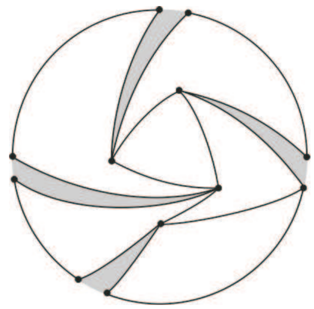

FIGURE 7. Another pointed rigidity circuit

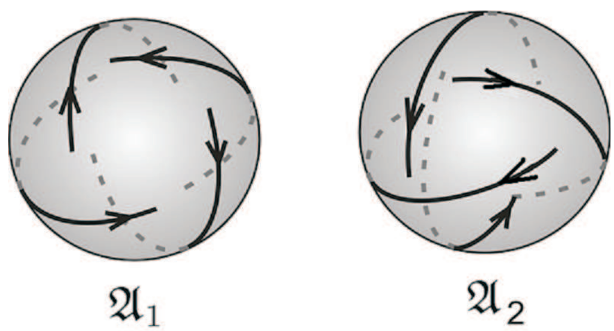


FIGURE 8. Two non-isotopic configurations of great circles

each of the pseudo-di-gons of $\mathcal{T}(G)$. This yields a configuration of four disjoint great semicircles on $S^2$.

The Example 4.5 and Example 4.7 give configurations from Fig. 8. (the first one and the second one respectively). These configurations are known to be non-isotopic (see [14]), i.e., there is no continuous motion which brings one of them to another avoiding crossings.

These different examples have yielded examples of non-isotopic hyperbolic hérissons (discussed in [13] and [5]). We recall that the existence of just one such surface was an open problem for a long time. The existence of the second isotopy type was a new surprise. In the framework of the present paper, we construct it easily.

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