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Homotopy coherent category theory and A_∞-structures in monoidal categories

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Abstract

We consider the theory of operads and their algebras in enriched category theory. We introduce the notion of simplicial A_∞-graph and show that some important constructions of homotopy coherent category theory lead by a natural way to the use of such objects as the appropriate homotopy coherent counterparts of the categories. © 1998 Elsevier Science B.V.

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0. Introduction

Let Λ be a small simplicial category, and let $F, G : \Lambda \to K$ be two simplicial functors to a simplicial category K. Then we can consider, as the simplicial set of coherent natural transformations from F to G, the coherent end [8, 10, 12] (see Definition 6.2):

$$\underline{Coh}(F,G) = \oint_\Lambda \underline{K}(F(\lambda), G(\lambda)).$$

It is well known that in general we cannot compose this transformation. A composition is defined only under some additional conditions [2, 10, 12] and is associative only up to homotopy. So if we want to obtain a category structure on the set of coherent transformations we have to pass to the homotopy classes of these transformations. The resulting category is no longer simplicial, or more precisely has a trivial simplicial structure.

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This phenomenon is of very general nature. It is sufficient to note that in the theory of simplicial closed categories of Quillen [22] the passage to the homotopy category $Ho(C)$ leads us to the losing of the initial simplicial structure.

Our attention to this phenomenon was motivated by the problem formulated by Bourn, Cordier and Porter, concerning categorical interpretation of strong shape theory [6, 11, 12]. Roughly speaking, the problem is to define the strong shape category for any simplicial functor and to compare such categories under the appropriate conditions.

Here the situation is analogous to that just described. The primary source of difficulties lies in the fact that we cannot construct any reasonable comparison functor between strong shape categories, because the passage to the homotopy classes leads to the losing of initial enriched structure. So we have to find some structure on the set of strong shape morphisms, which could play the role of simplicial structure on the category. Of course, we have to generalize also the notion of simplicial functor and to demand that this "functor" is isomorphism provided it is bijective on the objects and induces the homotopy equivalences of the "spaces of morphisms".

There is an immediate analogy between the problems above and the problems arising in the loop spaces theory and delooping machines [5, 20, 25, 27]. The purpose of this paper is to reveal this analogy and to provide some technical tools for constructing and working with this generalized category theory. As a line of attack we choose the theory of A_∞-operads and their algebras developed by May [20].

Our main idea is to replace the notions of simplicial category and simplicial functor by the notions of simplicial A_∞-graph and coherent morphism between A_∞-graphs. In some sense our paper extends the article [12], which considers the phenomenon of homotopy coherence on the level of coherent diagrams and their natural transformations. It is our opinion that if we want to obtain a closed variant of homotopy coherent category theory, we have to replace the categories by their homotopy coherent counterparts. In support of this statement we prove that many of the "homotopy coherent categories" are homotopy categories of some locally Kan A_∞-graphs. In particular, the "H-category" structure on the set of coherent transformations introduced in [10, 12] may be extended to an A_∞-structure (Theorem 6.3), at least for the case when K is a locally Kan category.

It should be pointed out that our approach is not the only possible. Another idea, which goes back to Boardman and Vogt [5, 29], is to use weak Kan complexes as the appropriate generalization of the notion of category. The reason is that in such a complex one can define the "compositions" of the chains of the 1-simplices, which is associative up to all higher homotopies. This notion lies at the heart of the Cordier–Porter methods. Recently it was used successfully by Günther [15] in strong shape theory.

We should mention also a paper [23] of Schwänzl and Vogt, where the ideas related to Segal' delooping machine were used.

Each of these methods has its own advantages and disadvantages, which we do not discuss here. Note only that our approach allows to call on the powerful methods of categorical algebra, whereas the alternative ones use the methods of homotopy theory.

We do not consider here the applications of our theory to the strong shape theory. It is the theme of a separate article [3].

Some words about the notations and terminology. If $\mathscr{A}$ is a monoidal category and K is a category enriched in $\mathscr{A}$ then $\underline{K}$ will mean the $\mathscr{A}$-enriched *hom*-functor in K [16].

An additional point to emphasize is that we must take some extra care concerning the size (in the set theoretical sense) of the objects under consideration. In our paper we are taking the view of Kelly on the existence of $\mathscr{A}$-categories of $\mathscr{A}$-functors (see the discussion on pp. 69–70 in [16]).

We now will undertake a brief overview of the paper.

Section 1 is devoted to the basic notions and definitions. We have to generalize May's initial approach in such a way that it becomes applicable in some enriched monoidal categories. Although it is sufficient for our applications to consider the topological and simplicial categories, we prefer to work in maximal generality having in view further applications to the categories enriched in the other monoidal categories (such as categories of groupoids, of categories and various categories of algebraic nature).

In our paper we concentrate attention on the monoidal categories, which are not symmetric. So the permutations may be disregarded in the theory. Thus our notion of operad is an immediate generalization of May's non Σ-operad. However, there is some difference, because we work without base points. So we do not require the triviality of the 0-term of our operads. This difference affects also the construction of a monad corresponding to an operad. It differs from that of May and agrees with the monad $\hat{C}^+$ introduced by Thomason in [27].

In Section 2 we construct a homotopy theory of algebras over an operad and consider the problem of "rectification" with respect to A_∞-monoid structure. The main results here are Theorems 2.3, 2.4 and Corollary 2.3.1 giving us the conditions under which an A_∞-monoid is isomorphic in some homotopy coherent sense to a monoid.

In Sections 3 we develop some technique of lifting of algebra structures with respect to a map of operads. We prove here that there is some universal A_∞-operad, which acts on every A_∞-monoid. In Section 4 we prove a variant of the theorem of homotopy invariance of the A_∞-monoid structure. The variants of the results of Sections 3 and 4 are well known in the theory of loop spaces [5, 17, 25].

In Section 5 we establish two main technical results, which give us a powerful method for constructing the A_∞-monoids in simplicial and topological monoidal categories. The first of these results (Theorem 5.1) asserts that the standard cosimplicial topological space Δ has a structure of an A_∞-comonoid. This A_∞-comonoid structure is a generalization of the rule of composition of higher homotopies considered in [2, 9, 18]. The second (Theorem 5.2) shows that the cosimplicial realization of a cosimplicial A_∞-monoid with respect to a cosimplicial A_∞-comonoid is again an A_∞-monoid.

In Section 6 we consider some applications of the results obtained to the homotopy coherent category theory. Our main goal here was to demonstrate that the A_∞-structures arise in a natural way in many important fields of category theory such as bar and cobar constructions and the above-mentioned problem of composition of coherent transformations.

1. Operads and their algebras in enriched monoidal categories

Let $\mathscr{A}$ be a closed symmetric monoidal category with tensor product $\otimes_{\mathscr{A}}$, unit object I and with initial object $\emptyset$ and let $\mathscr{A}$ be complete and cocomplete [16]. We denote by $\underline{\mathscr{A}}$ the internal *hom*-functor on $\mathscr{A}$. Let $\mathscr{N}$ be the set of natural numbers.

Definition 1.1. We call the category of $\mathscr{A}$-collections the $\mathscr{A}$-category of $\mathscr{A}$-functors from $\mathscr{N}$ (with discrete category structure) to $\mathscr{A}$. We shall denote it by *Coll*.

So the objects of *Coll* are the sequences of the objects of $\mathscr{A}$, and the enriched *hom*-functor from $A = (A_0, A_1, \ldots)$ to $B = (B_0, B_1, \ldots)$ is

$$\underline{Coll}(A,B) = \prod_{i=0}^{\infty} \underline{\mathscr{A}}(A_i, B_i).$$

Definition 1.2. Let K be an $\mathscr{A}$-category with a bifunctor $\otimes_K : K \otimes_{\mathscr{A}} K \to K$, an object I_K of K and isomorphisms

$$a_{A,B,C} : (A \otimes_K B) \otimes_K C \to A \otimes_K (B \otimes_K C),$$

$$l_A : I_K \otimes_K A \to A, r_A : A \otimes_K I_K \to A,$$

which make K a monoidal category [16]. We say that K is a monoidal $\mathscr{A}$-category if $\otimes_K$ is an $\mathscr{A}$-functor and a, l, r are $\mathscr{A}$-natural.

For example, the category *Coll* has the two following monoidal structures:

1. The fibrewise monoidal structure with fibrewise tensor product, which we shall denote $\otimes_{\mathscr{A}}$, as well, and with $\mathscr{M} = (I, I, \ldots)$ as a unit object.
2. For two collections A and B put

$$(A \otimes_{Coll} B)_j = \coprod_{k \geq 1} \coprod_{j_1 + \cdots + j_k = j} A_{j_1} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} A_{j_k} \otimes_{\mathscr{A}} B_k$$

for $j \geq 1$, and

$$(A \otimes_{Coll} B)_0 = B_0 \coprod \coprod_{k \geq 1} \underbrace{A_0 \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} A_0}_{k} \otimes_{\mathscr{A}} B_k.$$

Let I_{Coll} be a collection $(\emptyset, I, \emptyset, \ldots)$.

Remark. A similar tensor product of the collections was introduced by Smirnov in [24]. The basic difference is that our notions of collection and tensor product involve the 0-dimension terms.

Proposition 1.1. *The category Coll with $\otimes_{Coll}$ and I_{Coll} above is closed on the right monoidal $\mathscr{A}$-category with internal hom-functor*

$$\overline{Coll} : Coll^{op} \otimes_{\mathscr{A}} Coll \to Coll,$$

$$\overline{Coll}(A,B)_k = \prod_{j\geq 0} \prod_{j_1+\cdots+j_k=j} \underline{\mathscr{A}}(A_{j_1} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} A_{j_k}, B_j).$$

(That is

$$\underline{Coll}(A \otimes_{Coll} B, C) \simeq \underline{Coll}(B, \overline{Coll}(A,C)), \quad \forall A,B,C \in Ob(Coll).)$$

In addition, there is an $\mathscr{A}$-natural transformation

$$(A \otimes_{\mathscr{A}} B) \otimes_{Coll} (C \otimes_{\mathscr{A}} D) \to (A \otimes_{Coll} C) \otimes_{\mathscr{A}} (B \otimes_{Coll} D)$$

and a morphism $I_{Coll} \to \mathscr{M}$, which make $\otimes_{\mathscr{A}}$ a monoidal functor [13, 5].

Proof. The proof consists in immediate verification. □

Further we understand the category $Coll$ as a monoidal category with respect to the second structure and denote $\otimes_{Coll}$ and I_{Coll} by $\otimes$ and I respectively if it leads to no confusion.

Definition 1.3. We call an $\mathscr{A}$-operad $\mathscr{E}$ with multiplication

$$\gamma : \mathscr{E} \otimes \mathscr{E} \to \mathscr{E}$$

and unit

$$\eta : I \to \mathscr{E}$$

a monoid $(\mathscr{E}, \gamma, \eta)$ [19] in $Coll$.

We shall denote by $Oper$ the $\mathscr{A}$-category of operads.

Examples. 1. For any $\mathscr{A}$ the collection $\mathscr{M} = (I, I, \ldots)$ has a natural $\mathscr{A}$-operad structure.

2. Let K be a monoidal $\mathscr{A}$-category with enriched *hom*-functor $\underline{K}$, tensor product $\otimes_K$ and unit object I_K. For $X \in Ob(K)$ put

$$X^j = \underbrace{X \otimes_K \cdots \otimes_K X}_{j}, \quad j \neq 0, \qquad X^0 = I_K.$$

Then one can associate with X two $\mathscr{A}$-operads $End(X)$, $End^o(X)$:

$$End(X)_j = \underline{K}(X^j, X), \quad End^o(X)_j = \underline{K}(X, X^j)$$

called the operad of endomorphisms of X and dual operad of endomorphisms of X.

For some technical purposes we need the following generalization of the example 2 above. In the situation of the example consider two $\mathscr{A}$-functors:

$$\overline{\underline{K}}, \overline{\underline{K}}^o : K^{op} \otimes_{\mathscr{A}} K \to Coll,$$

$$\overline{\underline{K}}(X,Y)_j = \underline{K}(X^j,Y), \quad \overline{\underline{K}}^o(X,Y)_j = \underline{K}(X,Y^j).$$

There are also two obvious natural morphisms:

$$\mu : \overline{\underline{K}}(X,Y) \otimes \overline{\underline{K}}(Y,Z) \to \overline{\underline{K}}(X,Z),$$

$$\mu^o : \overline{\underline{K}}^o(X,Y) \otimes \overline{\underline{K}}^o(Y,Z) \to \overline{\underline{K}}^o(X,Z).$$

In addition, for each object X of K we have two morphisms in $Coll$:

$$\varepsilon : I \to \overline{\underline{K}}(X,X), \quad \varepsilon^o : I \to \overline{\underline{K}}^o(X,X)$$

generated by an $\mathscr{A}$-enriched structure on K.

Proposition 1.2. *The functors $\overline{\underline{K}}$ ($\overline{\underline{K}}^o$), natural transformation μ (μ^o) and morphism ε (ε^o), define some enrichment of K in Coll.*

Proof. Immediate from the definitions. □

Let $\mathscr{E}$ be an $\mathscr{A}$-operad.

Definition 1.4. An $\mathscr{E}$-algebra in K is a pair (X,η), where X is an object of K and η is a morphism of $\mathscr{A}$-operads

$$\eta : \mathscr{E} \to End(X).$$

So our definition of an algebra of operad is the same as in [20].

Definition 1.5 (*Smirnov* [24]). An $\mathscr{E}$-coalgebra in K is a pair (X,ρ), where X is an object of K and ρ is a morphism of $\mathscr{A}$-operads

$$\rho : \mathscr{E} \to End^o(X).$$

Usually, we write X for an $\mathscr{E}$-algebra ($\mathscr{E}$-coalgebra) (X,η) assuming that we know the structure map η. We write also $Alg^{\mathscr{E}}$ and $Coalg^{\mathscr{E}}$ for the $\mathscr{A}$-categories of $\mathscr{E}$-algebras and $\mathscr{E}$-coalgebras correspondingly.

Example. For any monoidal $\mathscr{A}$-category K the category of $\mathscr{M}$-algebras is isomorphic to the category of monoids in K.

Let K be a monoidal $\mathscr{A}$-category and let K be $\mathscr{A}$-tensored ($\mathscr{A}$-cotensored) by

$$(-) \otimes_{\mathscr{A}} (-) : K \otimes_{\mathscr{A}} \mathscr{A} \to K \;\; ((-)^{(-)} : \mathscr{A}^{op} \otimes_{\mathscr{A}} K \to K).$$

In general, there is no connection between this functor and tensor product $\otimes_K$ in K. But it will be convenient for us to assume that $(-)\otimes_{\mathscr{A}}(-)((-)^{(-)})$ is monoidal. That is for such a tensorization (cotensorization) we have an $\mathscr{A}$-natural transformation

$$\gamma : (X \otimes_{\mathscr{A}} A) \otimes_K (Y \otimes_{\mathscr{A}} B) \to (X \otimes_K Y) \otimes_{\mathscr{A}} (A \otimes_{\mathscr{A}} B),$$

$$(\omega : X^A \otimes_K Y^B \to (X \otimes_K Y)^{A \otimes_{\mathscr{A}} B})$$

and a morphism

$$\gamma_I : I_K \to I_K \otimes_{\mathscr{A}} I \;\; (\omega_I : I_K \to I_K^I),$$

which satisfy the axioms of monoidality [13, 5].

We shall say also that K is strongly $\mathscr{A}$-tensored ($\mathscr{A}$-cotensored) provided γ and γ_I (ω and ω_I) are isomorphisms.

Proposition 1.3. *Let K be a monoidal $\mathscr{A}$-category with coproducts, commuting with tensor product. If K is $\mathscr{A}$-tensored, then the formula*

$$[\mathscr{E}](X) = \coprod_{n \geq 0} (X^n) \otimes_{\mathscr{A}} \mathscr{E}_n, \quad \mathscr{E} \in Ob(Coll), \;\; X \in Ob(K)$$

defines an $\mathscr{A}$-functor from $Coll \otimes_{\mathscr{A}} K$ to K.

If, in addition, $\mathscr{E}$ is an $\mathscr{A}$-operad then the functor

$$[\mathscr{E}](-) : K \to K$$

has a natural structure of the $\mathscr{A}$-monad. The category of algebras of this monad is isomorphic to the category of $\mathscr{E}$-algebras. Furthermore, if K is strongly $\mathscr{A}$-tensored, then $[-](-) : Coll \otimes_{\mathscr{A}} K \to K$ provides the Coll-tensorization of the Coll-enrichment $\underline{K}$ of K.

Proof. The proof is well known [20]. □

Examples. 1. Let K be a monoidal $\mathscr{A}$-category with coproducts and zero object $\emptyset$ (that is $G \otimes_K \emptyset \simeq \emptyset \otimes_K G \simeq \emptyset$ for any $G \in Ob(K)$) and let $\otimes_K$ commute with coproducts. The category $Gr(C)$ of K-graphs over a fixed set of objects C is the following $\mathscr{A}$-category. The objects of $Gr(C)$ are the functions:

$$G : C \times C \to Ob(K).$$

The enriched *hom*-functor is

$$\underline{Gr(C)}(F, G) = \prod_{a,b \in C} \underline{K}(F(a,b), G(a,b)).$$

The monoidal structure is given by

$$(G \otimes_{Gr(C)} F)(a,b) = \coprod_{c \in C} G(a,c) \otimes_K F(c,b)$$

and

$$I_{Gr(C)}(a,b) = I \text{ if } a = b, \quad I_{Gr(C)}(a,b) = \emptyset \text{ if } a \neq b.$$

It is clear that $Gr(C)$ is strongly $\mathscr{A}$-tensored and $\mathscr{A}$-cotensored if K is. Note that the category of monoids in the category of $\mathscr{A}$-graphs is isomorphic to the category of $\mathscr{A}$-categories with fixed set of objects C and $\mathscr{A}$-functors between them, which are identities on C.

2. If C is an $\mathscr{A}$-category, then the category of $\mathscr{A}$-endofunctors $F(C,C)$ is monoidal $\mathscr{A}$-category with respect to the composition. It is $\mathscr{A}$-tensored ($\mathscr{A}$-cotensored) if C is $\mathscr{A}$-tensored ($\mathscr{A}$-cotensored). The category of monoids in $F(C,C)$ is isomorphic to the category of $\mathscr{A}$-monads on C.

3. Let C be an $\mathscr{A}$-category. Then we can consider the $\mathscr{A}$-category $Dist(C,C)$ of $\mathscr{A}$-distributors (or profunctors) from C to C [4], that is the $\mathscr{A}$-category of $\mathscr{A}$-functors

$$G : C^{op} \otimes_{\mathscr{A}} C \to \mathscr{A}.$$

The tensor product is defined by the coend:

$$(G \otimes_{Dist(C,C)} F)(X,Y) = \int^{C} G(X,C) \otimes_{\mathscr{A}} F(C,Y)$$

and unit object is

$$I_{Dist(C,C)} = \underline{C} : C^{op} \otimes_{\mathscr{A}} C \to \mathscr{A}.$$

As in the example 1 $Dist(C.C)$ is strongly $\mathscr{A}$-tensored and $\mathscr{A}$-cotensored.

It is clear that the example 1 is a special case of 3 and the functor of discretization $(-)^d$ induces a monoidal $\mathscr{A}$-functor:

$$Dist(C,C) \to Dist(C^d,C^d) \simeq Gr(Ob(C)).$$

We have also a fully faithful strict monoidal $\mathscr{A}$-embedding [4, 6, 11]:

$$\phi_* : F(C,C) \to Dist(C,C), \quad \phi_L(X,Y) = \underline{C}(X,LY), \; L \in Ob(F(C,C)).$$

So the example 2 is connected closely with 3 as well.

There is also another embedding [4, 6, 11]:

$$\phi^* : F(C,C)^{op} \to Dist(C,C), \quad \phi^L(X,Y) = \underline{C}(LX,Y), \; L \in Ob(F(C,C)),$$

with similar properties. This provides us with the following generalization of a Bousfield–Kan lemma [7, p. 27; 2, p. 283].

Proposition 1.4. *Let $\mathscr{E}$ be an $\mathscr{A}$-operad. Then an endofunctor $R : C \to C$ ($L : C \to C$) has an $\mathscr{E}$-algebra ($\mathscr{E}$-coalgebra) structure if and only if the distributor ϕ_R (ϕ^L) has this structure.*

4. More generally, let K be a cocomplete monoidal $\mathscr{A}$-category with strong $\mathscr{A}$-tensorization, and let $\otimes_K$ commute with colimits. If C is an $\mathscr{A}$-category, then we can

consider the following category of K-distributors $Dist_K(C,C)$. The objects are the $\mathscr{A}$-functors from $C^{op} \otimes_{\mathscr{A}} C$ to K. The morphisms are their $\mathscr{A}$-natural transformations. The unit object is defined by

$$I_{Dist_K(C,C)}(X,Y) = I_K \otimes_{\mathscr{A}} \underline{C}(X,Y).$$

Tensor product is defined as in example 3 with $\otimes_K$ instead of $\otimes_{\mathscr{A}}$.

Below we assume that the category $\mathscr{A}$ has a structure of a closed model category in the sense of Quillen [22].

Examples. 1. The cartesian monoidal categories:
- of simplicial sets $\mathscr{S}$,
- of compactly generated topological spaces $\mathscr{T}op$ [26],
- of (small) categories $\mathscr{C}at$,
- of groupoids Gpd.

2. The various categories of algebraic nature (see [14, 24]).
In this case the category $Coll$ has an obvious fibrewise closed model category structure.

Definition 1.6. A morphism of $\mathscr{A}$-operads is a local weak equivalence, if it is a weak equivalence of the corresponding $\mathscr{A}$-collections.

Definition 1.7. An $\mathscr{A}$-operad $\mathscr{E}$ together with a morphism $\mathscr{E} \to \mathscr{M}$ of $\mathscr{A}$-operads (augmentation), which is a locally weak equivalence, is an A_∞-operad. A morphism of A_∞-operads is a morphism of operads commuting with augmentations.

Definition 1.8. We say that an $\mathscr{E}$-algebra ($\mathscr{E}$-coalgebra) X in K is an A_∞-monoid (A_∞-comonoid) if $\mathscr{E}$ is an A_∞-operad.

In the category of $\mathscr{A}$-graphs we give a separate definition.

Definition 1.9. An A_∞-graph is an A_∞-monoid in $Gr(C)$.

2. Coherent homotopy theory of $\mathscr{E}$-algebras

In this section we assume that the category $\mathscr{A}$ has also a simplicial enrichment $\underline{\mathscr{A}_{\mathscr{S}}}$ such that

$$\underline{\mathscr{A}_{\mathscr{S}}}(X,Y) \simeq \underline{\mathscr{A}_{\mathscr{S}}}(I_{\mathscr{A}}, \underline{\mathscr{A}}(X,Y))$$

for any $X, Y \in Ob(\mathscr{A})$ and such that it is the monoidal $\mathscr{S}$-category with strong finite $\mathscr{S}$-tensorization $- \times -$. Furthermore, the simplicial category $\mathscr{A}_{\mathscr{S}}$ has a structure of simplicial closed model Quillen category [22]. The categories from the example 1 of the previous section are of this type.

Definition 2.1. An $\mathscr{A}$-category K is locally fibrant if $K(X,Y)$ is a fibrant object of $\mathscr{A}$ for all $X,Y \in Ob(K)$.

For a simplicial category C we denote by $\pi(C)$ its homotopy category, that is $\pi(C)$ has the same objects as C and

$$\pi(C)(X,Y) = \pi_0(\underline{C}(X,Y)).$$

For any $\mathscr{A}$-category K we can define an "underlying" $\mathscr{S}$-category $K_{\mathscr{S}}$ by

$$\underline{K}_{\mathscr{S}}(X,Y) = \underline{\mathscr{A}}_{\mathscr{S}}(I_{\mathscr{A}}, \underline{K}(X,Y)).$$

If, in addition, K is $\mathscr{A}$-tensored by $- \otimes_{\mathscr{A}} -$, then $K_{\mathscr{S}}$ is finitely $\mathscr{S}$-tensored by

$$X \times k = X \otimes_{\mathscr{A}} (I_{\mathscr{A}} \times k)$$

for a finite simplicial set k. We shall denote by $\pi(K)$ the category $\pi(K_{\mathscr{S}})$.

Lemma 2.1. *If the unit object of $\mathscr{A}$ is cofibrant then for any locally fibrant category K the underlying $\mathscr{S}$-category $K_{\mathscr{S}}$ is a locally Kan $\mathscr{S}$-category.*

Recall now a construction from [2]. A generalization of this construction will be considered in Section 6.

Let (L,μ,ε) be an $\mathscr{A}$-comonad on K. Then for every $X \in ob(K)$ one can define a simplicial object $L_*(X)$ of K putting

$$L_n(X) = L^{n+1}(X), \quad d_i = L^{n-i} \cdot \varepsilon \cdot L^i, \quad s_i = L^{n-i} \cdot \mu \cdot L^i. \tag{1}$$

Suppose, now, that $K_{\mathscr{S}}$ is finite $\mathscr{S}$-tensored and there exists the realization

$$L_\infty(X) = \int^n L_n(X) \times \Delta^n.$$

We thus obtain an $\mathscr{S}$-endofunctor on $K_{\mathscr{S}}$ which we shall denote by L_∞. The counit ε generates an $\mathscr{S}$-natural transformation

$$\varepsilon_\infty : L_\infty \to I.$$

By applying the functor $\underline{K}_{\mathscr{S}}(-,Y)$ to $L_*(Y)$ levelwise we obtain a cosimplicial object $\underline{K}_{\mathscr{S}}(L_*(X),Y)$ in $\mathscr{S}$ and

$$\underline{K}_{\mathscr{S}}(L_\infty(X),Y) \simeq Tot(\underline{K}_{\mathscr{S}}(L_*(X),Y)),$$

where Tot is the Bousfield–Kan total space functor [7].

Let K' be a full subcategory of K. As was proved in [2] (see also Section 6), if $\underline{K}_{\mathscr{S}}(L_*(X),Y))$ is a fibrant cosimplicial simplicial set in the sense of [7] for every $X,Y \in ob(K')$, then one can define a category $CHL_\infty - K'$ called the coherent homotopy category of the comonad (L,μ,ε), which has the same objects as K' and

$$CHL_\infty - K'(X,Y) = \pi_0(Tot(\underline{K}_{\mathscr{S}}(L_*(X),Y))).$$

If, in addition, there exists L_∞ in K', then one can define a simplicial transformation

$$\mu_\infty : L_\infty \to L_\infty^2,$$

such that $(L_\infty, \mu_\infty, \varepsilon_\infty)$ becomes a comonad on $\pi(K')$, and $CHL_\infty - K'$ is isomorphic to the Kleisli category of this comonad.

Let now $\mathscr{E}$ be an $\mathscr{A}$-operad. Let K be a monoidal $\mathscr{A}$-category with $\mathscr{A}$-tensorization. Then the monad $[\mathscr{E}]$ exists and the category of $\mathscr{E}$-algebras is isomorphic to the category of algebras of $[\mathscr{E}]$. So we have a pair of $\mathscr{A}$-adjoint functors :

$$T = [\mathscr{E}] : K \to Alg^{\mathscr{E}}, \quad U : Alg^{\mathscr{E}} \to K,$$

which generates by the usual way the monad on K and the comonad TU

$$\eta : TU \to I, \quad \rho : TU \to TUTU$$

on $Alg^{\mathscr{E}}$ that we shall denote by $[\mathscr{E}]$ as well. So we have a simplicial object $[\mathscr{E}]_*X$ in $Alg^{\mathscr{E}}$ for each $\mathscr{E}$-algebra X, which is none other then the May bar-construction $B_*([\mathscr{E}],[\mathscr{E}],X)$ [20].

Our first goal is to find the conditions under which $\underline{Alg^{\mathscr{E}}}_{\mathscr{S}}([\mathscr{E}]_*X, Y)$ is fibrant in the category of cosimplicial simplicial sets $c\mathscr{S}$.

For this we need a definition from [21].

Definition 2.2. Let $\mathscr{B}$ be a closed model Quillen category. Let, in addition, $\mathscr{B}$ be a monoidal category with respect to tensor product $\otimes_{\mathscr{B}}$ and unit object $I_{\mathscr{B}}$. We say that a cofibration $f : A \to X$ is closed (with respect to $\otimes_{\mathscr{B}}$) if the following condition is satisfied:

For any cofibration

$$g : B \to Y$$

the canonical morphism

$$(X \otimes_{\mathscr{B}} B) \coprod_{A \otimes_{\mathscr{B}} B} (A \otimes_{\mathscr{B}} Y) \to X \otimes_{\mathscr{B}} Y$$

is a cofibration.

Note that in $\mathscr{S}$ and $\mathscr{T}\!op$ all cofibrations are closed. For $\mathscr{S}$ this may be established immediately, whereas for $\mathscr{T}\!op$ this is one of the results of Steenrod [26] (see also [20]). Note also that if a morphism $f : A \to B$ in $Coll$ is such that f_n is a closed cofibration for every $n \geq 0$, then it is a closed cofibration in $Coll$ (with respect to $\otimes_{Coll}$).

Let now $f_i : A_i \to B_i$, $i = 1,n$ be a family of morphisms in $\mathscr{B}$. Let

$$Z_i = B_1 \otimes_{\mathscr{B}} \cdots \otimes_{\mathscr{B}} B_{i-1} \otimes_{\mathscr{B}} A_i \otimes_{\mathscr{B}} B_{i+1} \cdots \otimes_{\mathscr{B}} B_n,$$

$$Z_{ij} = B_1 \otimes_{\mathscr{B}} \cdots \otimes_{\mathscr{B}} A_i \otimes_{\mathscr{B}} \cdots \otimes_{\mathscr{B}} A_j \otimes_{\mathscr{B}} \cdots \otimes_{\mathscr{B}} B_n, \quad i < j.$$

Then we can construct the following diagram :

$$\begin{tikzcd}[column sep=small] Z_1 \ \cdots & Z_i & \cdots & Z_j & \cdots \ Z_n \\ & & Z_{ij} \arrow[ul, "d^j"] \arrow[ur, "d^i"'] & & 1\leq i<j\leq n \ , \end{tikzcd}$$

where $d^i = id\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}f_i\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}id$. Let $D(f_1,\ldots,f_n)$ be the colimit of this diagram. Then the maps

$$id\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}f_i\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}id : Z_i \to B_1\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}B_n$$

induce a canonical morphism

$$d : D(f_1,\ldots,f_n) \to B_1\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}B_n.$$

Lemma 2.2. *If in $\mathscr{B}$ the tensor product $\otimes_{\mathscr{B}}$ commutes on the right with finite colimits then for any family of closed cofibrations $f_i : A_i \to B_i$, $i = 1,n$ the canonical morphism*

$$d : D(f_1,\ldots,f_n) \to B_1\otimes_{\mathscr{B}}\ldots\otimes_{\mathscr{B}}B_n$$

is a cofibration.

Proof. This follows from the inductive arguments since $D(f_1,\ldots,f_n)$ may be obtained as a colimit of the following diagram:

$$\begin{tikzcd} A_1\otimes_{\mathscr{B}}D(f_2,\ldots,f_n) \arrow[r, "id\otimes_{\mathscr{B}}d"] \arrow[d] & A_1\otimes_{\mathscr{B}}B_2\otimes_{\mathscr{B}}\cdots\otimes_{\mathscr{B}}B_n \\ B_1\otimes_{\mathscr{B}}D(f_2,\ldots,f_n) & \end{tikzcd} \quad \square$$

Definition 2.3. An $\mathscr{A}$-operad $\mathscr{E}$ is cofibrant provided the unit of $\mathscr{E}$

$$I \to \mathscr{E}$$

is a closed cofibration in *Coll*.

Theorem 2.1. *Let K be a cocomplete monoidal $\mathscr{A}$-category with strong $\mathscr{A}$-tensorization. Let $\mathscr{E}$ be a cofibrant $\mathscr{A}$-operad and let X and Y be two $\mathscr{E}$-algebras.*

Then the cosimplicial simplicial set

$$\underline{Alg^{\mathscr{E}}{}_{\mathscr{S}}}([\mathscr{E}]_*X, Y)$$

is fibrant, if $\overline{\underline{K}}(X,Y)$ is a fibrant $\mathscr{A}$-collection.

Proof. Let K,L be two $\mathscr{A}$-categories and let K be cocomplete. Let $S : K \to L$ be an $\mathscr{A}$-functor having the right $\mathscr{A}$-adjoint $T : L \to K$. Then we have an $\mathscr{A}$-comonad

$A = ST$ on L and an $\mathscr{A}$-monad $B = TS$ on K. For $X, Y \in Ob(L)$ consider a cosimplicial object in $\mathscr{A}$,

$$U^* = \underline{L}(A_*X, Y), \quad U^n = \underline{L}(A_{n+1}X, Y).$$

Let $M^n(U^*)$ be the limit of the diagram

$$\begin{tikzcd}[column sep=small] U_0^n \cdots & U_i^n \arrow[dr, "s^{j-1}"'] & \cdots & U_j^n \arrow[dl, "s^i"] \cdots U_n^n \\ & & U_{ij}^{n-1} & \end{tikzcd} \quad 0 \le i < j \le n.$$

Then the codegeneracy morphisms $s^i : U^{n+1} \to U^n$ induce a morphism

$$r^n : U^{n-1} \to M^n(U^*).$$

Let Z^* be an augmented cosimplicial object B^*TX, $Z^n = B^{n+1}TX$ with cofaces and codegenaracies generated by the monad B on K and let $D^n(Z^*)$, $n \ge 0$ be the colimit of the diagram

$$\begin{tikzcd}[column sep=small] Z_0^n \cdots & Z_i^n & \cdots & Z_j^n \cdots Z_{n-1}^n \\ & & Z_{ij}^{n-1} \arrow[ul, "d^{j-1}"] \arrow[u, "d^i"'] & \end{tikzcd} \quad 0 \le i < j \le n+1.$$

For $n = -1$ we put $D^{-1}(Z^*) = Z^{-1}$. Then the coface morphisms $d^i : Z_i^n \to Z^{n+1}$ induce a canonical morphism

$$p^n : D^n(Z^*) \to Z^{n+1}.$$

We have the following lemma:

Lemma 2.3. *The canonical morphism*

$$r^n : U^{n+1} \to M^n(U^*)$$

is the result of application of the functor $\underline{K}(-, TY)$ *to the morphism*

$$p^{n-1} : D^{n-1}(Z^*) \to Z^n.$$

Proof. The statement is evident in view of the isomorphisms,

$$U^n = \underline{L}(A^{n+1}X, Y) \simeq \underline{K}(B^nTX, TY) = \underline{K}(Z^{n-1}, TY),$$

and the fact that codegeneracies in U^* are induced by cofaces in Z^*. $\square$

Apply now this lemma to the pair of $\mathscr{A}$-adjoints

$$T = [\mathscr{E}] : K \to Alg^{\mathscr{E}}, \quad U : Alg^{\mathscr{E}} \to K.$$

So the morphism

$$r^n : M^n(\underline{Alg^{\mathscr{E}}}([\mathscr{E}]_* X, Y)) \to \underline{Alg^{\mathscr{E}}}([\mathscr{E}]^{n+2}X, Y)$$

is the result of application of $\underline{K}(-, UY)$ to

$$p^{n-1} : D^{n-1}([\mathscr{E}]^* UX) \to [\mathscr{E}]^{n+1}UX.$$

But

$$\underline{K}([\mathscr{E}]^{k+1}X, Y) \simeq \underline{Coll}(\mathscr{E}^{k+1}, \overline{\underline{K}}(X, Y)),$$

$$\underline{K}(D^{n-1}([\mathscr{E}]^* X), Y) \simeq \underline{Coll}(D^{n-1}(\mathscr{E}^*), \overline{\underline{K}}(X, Y))$$

and so r^n is the result of the application of $\underline{Coll}(-, \overline{\underline{K}}(X, Y))$ to

$$\delta^{n-1} : D^{n-1}\mathscr{E}^* \to \mathscr{E}^{n+1}.$$

The last is a cofibration in $Coll$ by Lemma 2.2. Now Theorem 2.1 follows from the fibrantness of $\overline{\underline{K}}(X, Y)$. $\square$

Let K be a cocomplete monoidal $\mathscr{A}$-category with strong $\mathscr{A}$-tensorization and let K' be its full locally fibrant subcategory. Let $\mathscr{E}$ be a cofibrant $\mathscr{A}$-operad. Theorem 2.1 allows us to give the following definition:

Definition 2.4. We call the coherent homotopy category of $\mathscr{E}$-algebras in K', $CH\,Alg^{\mathscr{E}}(K')$, the coherent homotopy category

$$CH[\mathscr{E}]_\infty - Alg^{\mathscr{E}}(K'),$$

where $Alg^{\mathscr{E}}(K')$ is the full subcategory of $Alg^{\mathscr{E}}$ generated by objects of K'.

If K itself is locally fibrant, then we write simply $CH\,Alg^{\mathscr{E}}$ for $CH\,Alg^{\mathscr{E}}(K)$.

Let now X_* be a simplicial object in K. If K is cocomplete, then we can consider the realization functor $|-|$:

$$|X_*| = \int^k X_k \times \Delta^k.$$

There exists an obvious natural morphism

$$|X_* \otimes_* Y_*| \to |X_*| \otimes_K |Y_*|,$$

where $\otimes_*$ is the fibrewise tensor product of simplicial objects. We shall say that $\otimes_K$ commutes with realization if it is an isomorphism.

Theorem 2.2. *The canonical functor*

$$P : \pi Alg^{\mathscr{E}} \to CH\,Alg^{\mathscr{E}}$$

inverts the morphisms of $\mathscr{E}$-algebras, which are the homotopy equivalences in K.

Furthermore, P is the localization functor with respect to the above class of the morphisms of $\mathscr{E}$-algebras, provided $K_{\mathscr{S}}$ is cocomplete and $\otimes_K$ commutes with realization.

Proof. The proof follows from Theorem 2.1 and the results of [2]. Indeed, if $\otimes_K$ commutes with realization, then for any simplicial object X_* in K there is an $\mathscr{S}$-natural isomorphism

$$|[\mathscr{E}]X_*| \simeq [\mathscr{E}]|X_*|.$$

Thus, the comonad $([\mathscr{E}]_\infty, \rho_\infty, \eta_\infty)$ is defined and we have

$$[\mathscr{E}]([\mathscr{E}]_\infty X) = [\mathscr{E}]|[\mathscr{E}]_* X| \simeq |[\mathscr{E}]([\mathscr{E}]_* X)|.$$

Hence, the arguments dual to that of [2, Theorem 5.2] give us the desired result. □

Remark. For an $\mathscr{E}$-algebra X, an $\mathscr{E}$-algebra $[\mathscr{E}]_\infty X$ is none other than bar-construction $B([\mathscr{E}],[\mathscr{E}],X)$ [20].

Corollary 2.2.1. *If $\otimes_K$ commutes with realization, then the canonical functor P has a left adjoint Q, which is fully faithful and on the object X is equal to $B([\mathscr{E}],[\mathscr{E}],X)$. The category $CH\,Alg^{\mathscr{E}}$ is the Kleisli category of the comonad on $\pi Alg^{\mathscr{E}}$ generated by this adjunction.*

Let now $f : \mathscr{E}_0 \to \mathscr{E}_1$ be a morphism between two cofibrant $\mathscr{A}$-operads. Then it induces an obvious functor

$$f^* : Alg^{\mathscr{E}_1} \to Alg^{\mathscr{E}_0}.$$

On the other hand, if K is cocomplete and $\otimes_K$ commutes with realization, then for every $\mathscr{E}_0$-algebra X one can consider the bar-construction $B([\mathscr{E}_1],[\mathscr{E}_0],X)$, thus obtaining a functor

$$f_* : Alg^{\mathscr{E}_0} \to Alg^{\mathscr{E}_1}.$$

Proposition 2.1. *The functors f^* and f_* induce the functors*

$$F^* : CH\,Alg^{\mathscr{E}_1} \to CH\,Alg^{\mathscr{E}_0}, \quad F_* : CH\,Alg^{\mathscr{E}_0} \to CH\,Alg^{\mathscr{E}_1},$$

such that F^* is right adjoint to F_*.

If, in addition, f is a locally weak equivalence, then F_*, F^* is a pair of inverse equivalences of categories.

Proof. This is an easy consequence of Corollary 2.2.1 and the well-known properties of bar-construction (see the proof of Proposition 1.17 in [27]). The only thing we have to demonstrate is that the cosimplicial simplicial sets $\underline{K}_{\mathscr{S}}(B_*([\mathscr{E}_1],[\mathscr{E}_0],X),Z)$ and $\underline{K}_{\mathscr{S}}(B_*([\mathscr{E}_0],[\mathscr{E}_0],X),Z)$ are fibrant for every $Z \in Ob(K)$. Indeed,

$$\underline{K}(B_n([\mathscr{E}_1],[\mathscr{E}_0],X),Z) \simeq \underline{K}([\mathscr{E}_1][\mathscr{E}_0]^n X,Z) \simeq \underline{Coll}([\mathscr{E}_1][\mathscr{E}_0]^n,\overline{\underline{K}}(X,Z)).$$

Thus the arguments used in the proof of Theorem 2.1 give the desired result. □

Applying this proposition to the augmentation of a cofibrant A_∞-operad $\mathscr{E}$, we see that the coherent homotopy category of $\mathscr{E}$-algebras is canonically equivalent to the coherent homotopy category of $\mathscr{M}$-algebras. Hence, for two cofibrant A_∞-operads $\mathscr{E}_0, \mathscr{E}_1$, there is a canonical equivalence of categories

$$F : CH\,Alg^{\mathscr{E}_0} \to CH\,Alg^{\mathscr{E}_1}.$$

So one can define for an $\mathscr{E}_0$-algebra X and an $\mathscr{E}_1$-algebra Y the set of their coherent morphisms as

$$CH\,Alg^{\mathscr{E}_1}(F(X),Y).$$

It is evident that we thus obtain a category $CH\,Mon$, with the algebras over different cofibrant A_∞-operads as objects, which we call the coherent homotopy category of A_∞-monoids. We summarize:

Theorem 2.3. *If $K_{\mathscr{S}}$ is cocomplete, locally fibrant and $\otimes_K$ commutes with realization, then the inclusion*

$$CH\,Alg^{\mathscr{M}} \subset CH\,Mon$$

is an equivalence of categories. In particular, every A_∞-monoid is isomorphic in $CH\,Mon$ to a honest monoid.

Remark. There exists an obvious forgetful functor

$$U : CH\,Mon \to \pi(K),$$

such that the inclusion $CH\,Alg^{\mathscr{M}} \subset CH\,Mon$ is a functor over $\pi(K)$. However, an inverse equivalence does not. In the following section we construct another A_∞-operad, for which the corresponding equivalences commute with U.

Corollary 2.3.1. *Let all the conditions of Theorem 2.3 be fulfilled for $\mathscr{A}$ itself. Then the category of $\mathscr{A}$-graphs over a fixed set of objects C satisfies these conditions too. Hence each A_∞-graph is isomorphic to some $\mathscr{A}$-category.*

This corollary shows that in many important cases an A_∞-graph admits a "rectification". It is true, for example, for the category of $\mathscr{T}\!op$-graphs. Unfortunately, we cannot apply it to the important category of $\mathcal{S}$-graphs. However, in this case we can slightly change the proof according to one idea of [12].

Namely, we shall say that an object X of K is fibrant provided the object $\underline{K}(A,X)$ of $\mathcal{A}$ is fibrant for every $A \in Ob(K)$. Let K_f denote the full subcategory of fibrant objects in K. Let, in addition, K be cocomplete and $\otimes_K$ commute with realization; then for every $\mathscr{E}$-algebra X in K_f we can consider the bar-construction $B(\mathscr{E},\mathscr{E},X)$, which is an $\mathscr{E}$-algebra in K. Then the coherent $CHAlg^{\mathscr{E}}(K_f)$ is defined correctly and every coherent morphism from X to Y may be specified up to homotopy as the morphism of $\mathscr{E}$-algebras in K:

$$B(\mathscr{E},\mathscr{E},X) \to Y.$$

The unit of the monad $[\mathscr{E}]$ gives us a morphism $\eta : X \to B(\mathscr{E},\mathscr{E},X)$ in K.

Lemma 2.4. *Let X and Y be two $\mathscr{E}$-algebras in K_f. And let a morphism of $\mathscr{E}$-algebras $\phi : B(\mathscr{E},\mathscr{E},X) \to Y$ be such that $\phi \cdot \eta$ is a homotopy equivalence. Then ϕ defines an isomorphism in the coherent homotopy category of $\mathscr{E}$-algebras in K_f.*

Proof. This is evident via Theorem 2.1 and the fact that η induces a deformation retraction

$$\underline{K}(X,Z) \leftarrow \underline{K}(B(\mathscr{E},\mathscr{E},X),Z)$$

for every fibrant object Z in K [20, 2, Lemma 2.1]. □

Return to the case of the category of $\mathcal{S}$-graphs. We shall say that $\mathcal{S}$-graph G is locally Kan if $G(a,b)$ is Kan for every $a,b \in C$. It is obvious, that the category of locally Kan graphs is the subcategory of fibrant objects in the category of simplicial graphs. If $f : \mathscr{E}_0 \to \mathscr{E}_1$ is a morphism between two $\mathcal{S}$-operads, then for every $\mathscr{E}_0$-algebra X in $Gr(C)$ one can consider a locally Kan graph

$$S|B([\mathscr{E}_1],[\mathscr{E}_0],X)| \simeq S(B([\mathscr{E}_1],[\mathscr{E}_0],|X|)),$$

where S is the fibrewise singular complex functor and $|-|$ is the functor of fibrewise geometric realization. But $S(B([\mathscr{E}_1],[\mathscr{E}_0],|X|))$ is, obviously, an $\mathscr{E}_1$-algebra. Moreover, if f is a locally weak equivalence and X is locally Kan, then the conditions of Lemma 2.4 are satisfied for the morphism

$$B([\mathscr{E}_0],[\mathscr{E}_0],X) \to B([\mathscr{E}_1],[\mathscr{E}_0],X) \to S(|B([\mathscr{E}_1],[\mathscr{E}_0],X)|).$$

Therefore, the conclusions of Proposition 2.1 and Theorem 2.3 remain valid. In particular we obtain:

Theorem 2.4. *Every simplicial locally Kan A_∞-graph is isomorphic in CHMon to a locally Kan simplicial category.*

3. Lifting of A_∞-structures

We shall use below some technique which goes back to [5] and is described in a form convenient for us in [14]. But some modification is necessary, because we want to work without permutations, but with homotopy units.

A tree is a nonempty finite connected planar (a 1-dimensional subspace of a plane) oriented graph T without loops, such that there is at least one incoming edge and exactly one outgoing edge at each vertex of T. Let $V(T), E(T)$ be the sets of vertices and of edges of T. We say that an edge e of T is external if it is bounded by a vertex at one end only and internal otherwise. The degenerate tree is a tree with a single edge and without vertices. The notions of output and input edges of a tree and of a vertex are the same as in [14]. Every tree has a single output edge, which is bounded by a vertex called the endpoint of the tree. We denote by $In(T), Out(T)$ the sets of input and output edges of a tree T and by $In(v), Out(v)$ the sets of input and output edges of a vertex $v \in V(T)$.

We fix some orientation on the plane and assume that the orientation of the edges of a tree T agrees with the direction of the x-axis. So we can consider an order on $In(T)$ (and on $In(v)$ for every $v \in V(T)$) according to the direction of the y-axis.

We denote by $[n]$ the set $\{1, 2, \ldots, n\}$ with the natural order. A tree T equipped with an order preserving injection $[n] \to In(T)$ will be referred to as an n-labelled tree or simply an n-tree. A 0-tree is a tree without labelling. A degenerate 0-tree will be denoted by D. A unit tree D^1 is a 1-labelled degenerate tree. A tree with a single vertex and n input edges is called an n-star. Two n-tree T, T' are called isomorphic if there exists an isomorphism of trees $T \to T'$ preserving orientations, the order on $In(v), v \in V(T)$ and labellings. Below we identify a labelled tree with its classes of isomorphism.

We say that e is a zero edge if it is an input edge without labelling. For any vertex $v \in V(T)$ we denote by $In'(v)$ the set of input nonzero edges at v.

As in [14] the composition $T_1 \circ_i T_2$ of labelled trees is defined. We consider also another type of composition. Let T be an n-tree and let $(T_v, v \in V(T))$ be a family of trees, such that T_v is an $In'(v)$-tree. Then one can define a new tree

$$T \circ_{v \in V(T)} T_v$$

called the composition of T and $(T_v, v \in V(T))$. By definition

$$V(T \circ_{v \in V(T)} T_v) = \bigcup_{v \in V(T)} V(T_v).$$

If $v \in V(T)$ is such that T_v is the unit or degenerate tree, then put $Z(v)$ to be the set of all input zero edges at v. For other v's put $Z(v) = \emptyset$. Then define

$$E(T \circ_{v \in V(T)} T_v) = \left(E(T) \Big\backslash \left(\bigcup_{v \in V(T)} Z(v) \right) \right) \cup \bigcup_{v \in V(T)} E(T_v) / \sim,$$

where $\sim$ is an equivalence generated by the following relations:

1. $Out(v) \sim Out(T_v)$.
2. The kth edge in $In'(T_v)$ is equivalent to the kth edge in $In'(v)$.

We put, in addition, that a zero edge in $In(v)$ is bounded in $T \circ_{v\in V(T)} T_v$ by the end point of T_v.

Let T be an n-tree and e be an edge of T. If e is an internal edge or an input zero edge bounded by a vertex v with $In(v) \neq e$ then we can form a new tree T/e by contracting e into a point (see [14]). We assume also that if a 0-tree T is isomorphic to a 1-star and e is a single input edge of T, then $T/e = D$. Analogously, if a 1-tree T is isomorphic to a 1-labelled 1-star and e is its single input edge, then $T/e = D^1$. In all the above cases we say that T/e is obtained from T by edge contraction.

Write $T \geq T'$ if T' is isomorphic to a tree obtained from T by a sequence of edge contractions. Thus $\geq$ is a partial order on the set of isomorphism classes of n-trees.

Let now $\mathscr{E}$ be an $\mathscr{A}$-collection and let S be a finite set with $|S| = k$. Then we put $\mathscr{E}_S = \mathscr{E}_k$. If now T is an n-tree we define

$$\tilde{\mathscr{E}}(T) = \bigotimes_{v\in V(T)} \mathscr{E}_{In'(v)} \tag{2}$$

(compare with formula (1.2.2) from [14]). We assume in addition, that $\tilde{\mathscr{E}}(D) = \tilde{\mathscr{E}}(D^1) = I$. This allows us to introduce the following $\mathscr{A}$-functor:

$$F : Coll \to Oper,$$

$$F(\mathscr{E})_n = \coprod_{T\in Tr(n)} \tilde{\mathscr{E}}(T),$$

where $Tr(n)$ is the set of isomorphism classes of n-trees. A multiplication in $F(\mathscr{E})$ is induced by the composition $\circ_i$ of trees [14] and a unit is defined by the canonical inclusion of I as a summand in $F(\mathscr{E})_1$ with unit tree as index. We have also a natural inclusion of $\mathscr{A}$-collections

$$\eta : \mathscr{E} \to F(\mathscr{E}),$$

where η_n is the canonical inclusion of the summand indexed by n-labelled n-star (for $n = 0$ this index is 1-star without labelling).

If $\mathscr{E}$ is an operad and $T \geq T'$, then one can define correctly a morphism

$$\tilde{\gamma}_{T,T'} : \tilde{\mathscr{E}}(T) \to \tilde{\mathscr{E}}(T')$$

as in [14], except for the case when $T' = D$ or $T' = D^1$. This allows us to define a natural morphism of operads

$$\gamma : F(\mathscr{E}) \to \mathscr{E}.$$

It is easy to see that F is a left adjoint to the forgetful functor

$$U : Oper \to Coll$$

with η as unit and γ as counit of the adjunction. This generates an $\mathscr{A}$-comonad (F,ρ,γ) on *Oper*.

Proposition 3.1. *Let $\mathscr{E}$ and $\mathscr{E}'$ be two operads, such that $\mathscr{E}$ is cofibrant and $\mathscr{E}'$ is fibrant as $\mathscr{A}$-collections. Then the cosimplicial simplicial set*

$$\underline{Oper}_{\mathscr{S}}(F_*(\mathscr{E}),\mathscr{E}')$$

is fibrant.

Proof. Let us give the following inductive definition. A 0-stage n-tree is an n-tree. A k-stage n-tree (or simply (k,n)-tree) $(T,T_v,v\in V(T))$ is a set consisting of an n-tree T and one $(k-1,In'(v))$-tree T_v for every vertex $v\in T$. Two (k,n)-trees $(T,T_v,v\in V(T))$ and $(T',T'_v,v\in V(T'))$ are called isomorphic if there exists a system consisting of isomorphism of n-trees $\phi: T\to T'$ and isomorphisms $\phi_v: T_v\to T'_{\phi(v)}$ of $(k-1,In'(v))$-trees. Let $Tr(k,n)$ be the set of isomorphism classes of (k,n)-trees. The sets $Tr(k,n)$, $k\geq 0$ make up a cosimplicial set. The coface operators are generated by operation of substitution of $In(v)$-labelled $In(v)$-stars at every vertex v of a tree T. The codegeneracy operators are induced by the composition of trees of the second type.

Let now $\mathscr{E}$ be an $\mathscr{A}$-collection and let $Q=(T,T_v,v\in V(T))$ be a (k,n)-tree. If $k=0$ then we define $\tilde{\mathscr{E}}(Q)$ by formula (2). By induction

$$\tilde{\mathscr{E}}(Q)=\bigotimes_{v\in V(T)}\tilde{\mathscr{E}}(T_v).$$

Then we have a natural isomorphism

$$F^k(\mathscr{E})_n\simeq\coprod_{T\in Tr(k-1,n)}\tilde{\mathscr{E}}(T).$$

Moreover, it is evident that the morphisms of codegeneracy and coface in $F^*(\mathscr{E})_n$ are induced by corresponding operators in $Tr(*,n)$. It implies that the morphism

$$p^k: D^k(F^*(\mathscr{E}))\to F^{k-1}(\mathscr{E})$$

is an isomorphism on the summand.

The proposition follows now from fibrantness of $\mathscr{E}'$ and Lemma 2.3. □

Let K be a cocomplete monoidal locally fibrant strongly $\mathscr{A}$-tensored $\mathscr{A}$-category, $\mathscr{E}$ be a cofibrant $\mathscr{A}$-operad and let $U: CHAlg^{\mathscr{E}}\to\pi(K)$ be a forgetful functor.

Definition 3.1. Let X be an object of K, and let

$$\eta,\nu:\mathscr{E}\to End(X)$$

be two $\mathscr{E}$-algebra structures on X. We say that they are coherently isomorphic provided there is an isomorphism r in $CHAlg^{\mathscr{E}}$ of (X,η) and (X,ν), such that $U(r)$ is identity.

Lemma 3.1. *Let* $(\mathscr{E},\mu,\varepsilon)$ *be a cofibrant* $\mathscr{A}$*-operad and let*

$$\eta, \nu : \mathscr{E} \to End(X)$$

be two $\mathscr{E}$*-algebra structures on X, such that* η *is homotopic to* ν *in* $Oper_{\mathscr{S}}$*. Then these structures are coherently isomorphic. This coherent isomorphism is not canonical and depends on the choice of homotopy class of homotopies between* η *and* ν*.*

Conversely, if μ *and* η *are coherently isomorphic, then they are homotopic as morphisms in* $Oper_{\mathscr{S}}$*.*

Proof. Let $X_0 = (X,\eta)$ and $X_1 = (X,\nu)$. For every $k \geq 1$ we have an isomorphism

$$\underline{Alg}^{\mathscr{E}}_{\mathscr{S}}([\mathscr{E}]^k X_0, X_1) \simeq \underline{K}_{\mathscr{S}}([\mathscr{E}]^{k-1}X_0, X_1).$$

Hence, to obtain a required isomorphism we have to construct some morphisms

$$r^k : ([\mathscr{E}]^k X) \times \Delta^k \to X, \quad k \geq 1,$$

$$r^0 = 1_X,$$

satisfying the coherency conditions. We shall do it by induction.

Let us denote by $\delta_n : \Delta^n \to \Delta^n \times \Delta^n$ the diagonal mapping. The iteration of δ_n generates a natural transformation

$$D_n : [\mathscr{E}]X \times \Delta^n \to [\mathscr{E}](X \times \Delta^n).$$

Note, that cofibrantness of $\mathscr{E}$ and fibrantness of the collection $End(X)$ imply the existence of a homotopy

$$r^1 : [\mathscr{E}]X \times \Delta^1 \to X,$$

for which

$$r^1(1_{[\mathscr{E}]X} \times d^0) = \eta, \tag{3}$$

$$r^1(1_{[\mathscr{E}]X} \times d^1) = \nu, \tag{4}$$

$$r^1(\varepsilon \times 1_{\Delta^1}) = 1_X \times s^0 \tag{5}$$

and the following diagram commutes:

$$\begin{tikzcd}
{[\mathscr{E}]^2X \times \Delta^1} \arrow[r, "1\times\delta_1"] \arrow[d, "\mu\times 1"'] & {[\mathscr{E}]^2X \times \Delta^1 \times \Delta^1} \arrow[r, "D_1\times 1"] & {[\mathscr{E}]([\mathscr{E}]X \times \Delta^1) \times \Delta^1} \arrow[d, "{[\mathscr{E}](r^1)\times 1}"] \\
{[\mathscr{E}]X \times \Delta^1} \arrow[r, "r^1"] & X & {[\mathscr{E}]X \times \Delta^1} \arrow[l, "r^1"']
\end{tikzcd}$$

Let now r^k be defined for $k \leq n-1$. Define r^n by the following composition:

$$([\mathscr{E}]^n X) \times \Delta^n \xrightarrow{s^0 \times (s^{n-1})^{n-1}} ([\mathscr{E}]^n X) \times \Delta^{n-1} \times \Delta^1 \xrightarrow{D_{n-1} \times 1}$$
$$\to [\mathscr{E}]([\mathscr{E}]^{n-1} X \times \Delta^{n-1}) \times \Delta^1 \xrightarrow{r^1([\mathscr{E}](r^{n-1} \times 1))} X.$$

It is not hard to verify, using formulas (3)–(5) and the diagram above, that the morphisms r^k define a desired coherent morphism.

The converse statement is evident. □

Let now the tensor product in $\mathscr{A}$ commute with realization. Then the realization of a simplicial operad in $\mathscr{A}$ is again an operad. So the bar-construction $B(F, F, \mathscr{E})$ is an $\mathscr{A}$-operad, which we shall denote by $F_\infty(\mathscr{E})$. Note, that $F_\infty(\mathscr{E})$ is a cofibrant A_∞-operad if $\mathscr{E}$ is.

Example. The operad $F_\infty(\mathscr{M})$ in $\mathscr{S}$ admits the following description. Let $n \geq 0$. Then $F_\infty(\mathscr{M})_n$ is the nerve of the partial ordered set of isomorphism classes of n-trees. The composition $\circ_i$ induces a multiplication by an obvious way.

The operad $F_\infty(\mathscr{M})$ contains some A_∞-suboperad $\mathscr{H}$ generated by reduced trees. More precisely, an n-tree T is reduced if there are at least two inputs at each vertex $v \in V(T)$. We assume also that D and D^1 are reduced, and that D is a single reduced 0-tree (compare with [14, 1.2.12]).

Let us give a description of $\mathscr{H}$ in terms of bracketing.

Let h_n be the following category:

(a) h_0 contains only one object 1 and only identity morphism;

(b) for $n > 0$ the objects of h_n are the sequences consisting of n formal symbols $a_1 \dots a_n$, several symbols 1 together with several pairs of brackets, such that the result of the corresponding formal product is meaningful (the repetition and the inverses of the generating elements $a_1, a_2, \dots$ are forbidden).

For example h_3 contains as the objects the following sequences:

$$a_1 a_2 a_3, \;\; 1a_1(a_2 1 1 a_3), \;\; (1(a_1 a_2)1)(a_3 a_1)$$

and so on. We have a morphism from a sequence A_0 to A_1 if A_0 may be obtained from A_1 by throwing off some pairs of brackets and some symbols 1.

We have the functors

$$\gamma : h_{j_1} \times \cdots \times h_{j_n} \times h_n \to h_{j_1 + \cdots + j_n},$$

which substitutes the object of h_{j_i} in place of the symbol a_i in k_n (in brackets) and renumbers the symbols in the natural order. There is also a functor $\varepsilon : 1 \to h_1$, which sends 1 to a_1. We thus have a $\mathscr{C}at$-operad h.

We obtain an $\mathscr{S}$-operad $\mathscr{H}$ by applying the nerve functor to h. The geometric realization of $\mathscr{H}$ gives us some topological A_∞-operad H, which is a deformation retract of $F_\infty(\mathscr{M})$.

The following result is a variant of a lifting theorem of [5].

Theorem 3.1. *Let $f : \mathscr{E} \to \mathscr{E}'$ be a local weak equivalence of cofibrant $\mathscr{A}$-operads. If in $\mathscr{A}$ the tensor product of weak equivalences is a weak equivalence, then for every $F_\infty(\mathscr{E})$-algebra structure η on X there exists an $F_\infty(\mathscr{E}')$-algebra structure v on X such that $(F_\infty(f)^*(v), X)$ is coherently isomorphic to (η, X) and any two such structures are coherently isomorphic.*

Proof. Under the conditions of the theorem we obtain a homotopy equivalence

$$\underline{Oper}_{\mathscr{S}}(F_k(\mathscr{E}'), End(X)) \to \underline{Oper}_{\mathscr{S}}(F_k(\mathscr{E}), End(X))$$

for every $k \geq 0$. Hence, $F_\infty(f)$ induces a homotopy equivalence

$$\underline{Oper}_{\mathscr{S}}(F_\infty(\mathscr{E}'), End(X)) \to \underline{Oper}_{\mathscr{S}}(F_\infty(\mathscr{E}), End(X))$$

by Proposition 3.1. Thus the theorem follows from Lemma 3.1. $\square$

Let $\mathscr{E}$ be a cofibrant A_∞-operad. By applying the theorem obtained to the augmentation of $\mathscr{E}$ we see that one can replace every $\mathscr{E}$-algebra structure on X by an $F_\infty(\mathscr{M})$-algebra structure on X without altering the isomorphism class of X in the coherent homotopy category of A_∞-monoids.

Thus we have

Corollary 3.1.1. *The natural inclusion*

$$CH\,Alg^{F_\infty(\mathscr{M})} \subset CH\,Mon$$

is an equivalence of categories over $\pi(K)$.

4. Homotopy invariance

The material of this section is sufficiently known and goes back to [5]. The detailed proof of the theorem below in terms of strong homotopy algebras over monads was given by Lada in [17]. We need, however, a slight generalization of his results, because we work with homotopy units. Furthermore, we do not use the notion of strong homotopy algebra. It is not hard to verify that for an operad $\mathscr{E}$ the structure on X of a strong homotopy algebra (with homotopy unit) over monad $[\mathscr{E}]$ is exactly an $F_\infty(\mathscr{E})$-algebra structure. Otherwise the methods of [17] work well and this allows us to minimize the volume of the proof.

Let $\mathscr{E}$ be an $\mathscr{A}$-operad, then we have a morphism of operads

$$\gamma_\infty : F_\infty(\mathscr{E}) \to \mathscr{E}.$$

Let (X, η) be an $\mathscr{E}$-algebra in some monoidal $\mathscr{A}$-category K and let $f : X \to Y$ be a homotopy equivalence in K.

Theorem 4.1. *If the collections $End(X)$ and $End(Y)$ are fibrant and $\mathscr{E}$ is a cofibrant operad, then f defines*
– *a structure of $F_\infty(\mathscr{E})$-algebra on Y,*

$$v : F_\infty(\mathscr{E}) \to End(Y),$$

– *an isomorphism,*

$$\phi : \gamma^*_\infty((X,\eta)) \to (Y,v),$$

in $CHAlg^{F_\infty(\mathscr{E})}$, such that $U(\phi) = [f]$, where U is the corresponding forgetful functor.

To prove this theorem, we need some lemma about homotopy equivalences, which goes back to Vogt [28] and is considered in general form in [10, Proposition 2.3] (see also [17, Lemma 9.1]). Our present form of this lemma may be easily deduced from it.

Lemma 4.1. *Let X, Y be two objects of some simplicial category K, such that $\underline{K}(X,X)$ and $\underline{K}(Y,Y)$ are Kan simplicial sets. Let*

$$f : X \to Y, \quad g : Y \to X$$

be the inverse homotopy equivalences and let

$$\xi_1 \in \underline{K}_1(X,X)$$

be a homotopy, such that $d_0(\xi_1) = 1_X$, $d_1(\xi_1) = g \cdot f$.
Then there exist a homotopy

$$\psi_1 \in \underline{K}_1(Y,Y)$$

and a 2-homotopy

$$\xi_2 \in \underline{K}_2(X,X),$$

such that

$$d_0(\psi_1) = f \cdot g, \quad d_1(\psi_1) = 1_Y,$$

$$d_0(\xi_2) = \xi_1 \cdot \xi_1,$$

$$d_1(\xi_2) = \xi_1,$$

$$d_2(\xi_2) = g \cdot \psi_1 \cdot f.$$

Proof of Theorem 4.1. To construct a desired $F_\infty(\mathscr{E})$-algebra structure on Y we show that the date of the theorem generates some morphism of operads

$$h : F_\infty(End(X)) \to End(Y).$$

Then one can define $v = h \cdot F_\infty(\eta)$.

As was shown in Lemma 3.1 we have to construct a sequence of morphisms of $\mathscr{A}$-collections,

$$h_n : F^n(End(X)) \times \Delta^n \to End(Y),$$

satisfying the coherence conditions. To do this we introduce some functor

$$F' : Coll \to Oper,$$

$$F(\mathscr{E})_n = \coprod_{T \in Tr(n), T \neq D^1} \tilde{\mathscr{E}}(T).$$

So $F(\mathscr{E}) \simeq F'(\mathscr{E}) \coprod I$.

Let us choose an inverse homotopy equivalence $g : Y \to X$ and a homotopy ξ_1 between 1_X and $g \cdot f$. By Lemma 4.1 we thus obtain a homotopy ψ_1 and a 2-homotopy ξ_2. We can consider the homotopies above as simplicial mappings

$$\xi_1 : \Delta^1 \to \underline{K}_{\mathscr{S}}(X,X), \quad \psi_1 : \Delta^1 \to \underline{K}_{\mathscr{S}}(Y,Y), \quad \xi_2 : \Delta^2 \to \underline{K}_{\mathscr{S}}(X,X)$$

respectively. This allows us to construct h_n by induction.

Let h_0 be the composition of the morphisms induced by f and g,

$$End(X) \xrightarrow{f_*} \underline{\overline{K}}(X,Y) \xrightarrow{g^*} End(Y).$$

To define h_n, $n \geq 1$ we need some auxiliary morphisms

$$k_n : F'(F^n End(X)) \times \Delta^2 \to F^n End(X), \quad n \geq 0.$$

Let k_0 be the following composition:

$$F'(End(X)) \times \Delta^2 \xrightarrow{1 \times proj.} F'(End(X)) \xrightarrow{\gamma} End(X).$$

Let $n \geq 1$, $k \geq 0$ and let T be a k-tree. Then the iterated diagonal $\Delta^2 \to \Delta^2 \times \Delta^2$ induces the following morphism:

$$\begin{aligned}
&\bigotimes_{v \in V(T)} (F^n End(X))_{In'(v)} \times \Delta^2 \simeq \\
&\qquad \bigotimes_{v \in V(T), v \neq Out(T)} (F^n End(X))_{In'(v)} \otimes (F^n End(X))_{In'(Out(v))} \times \Delta^2 \to \\
&\qquad \bigotimes_{v \in V(T), v \neq Out(T)} (F^n End(X) \times \Delta^2)_{In'(v)} \otimes (F^n End(X))_{In'(Out(v))}.
\end{aligned} \tag{6}$$

Let $v \in V(T), v \neq Out(T)$ then

$$(F^n End(X))_{In'(v)} = \coprod_{T' \in Tr(|In'(v)|)} F^{n-1} End(X)(T').$$

Define some morphism on the summand with index $T' \neq D^1$ by

$$
\begin{aligned}
(F^{n-1}End(X))(T') \times \Delta^2 &\xrightarrow{1 \times s_0(\xi_1)} (F^{n-1}End(X))(T') \times \underline{K}_{\mathscr{S}}(X,X) \\
&\longrightarrow (F^{n-1}End(X))(T'),
\end{aligned} \tag{7}
$$

where the last morphism is induced by multiplication in $End(X)$ (see the description of $(F^nEnd(X))_{In'(v)}$ in Proposition 3.1).

If $T' = D^1$, then define a morphism by the composition

$$
\begin{aligned}
(F^{n-1}End(X))(T') \times \Delta^2 &\xrightarrow{\xi_2} \underline{K}_{\mathscr{S}}(X,X) \times (F^{n-1}End(X))(T') \\
&\longrightarrow (F^{n-1}End(X))(T').
\end{aligned} \tag{8}
$$

The morphisms (7),(8) define together the morphisms

$$(F^nEnd(X))_{In'(v)} \times \Delta^2 \to (F^nEnd(X))_{In'(v)},$$

which in composition with (6) give the desired morphism k_n on the summand corresponding to a k-tree T.

Let now the morphisms h_m be defined for $m \leq n-1$. Then define h_n on the summand $F'(F^{n-1}End(X)) \times \Delta^n$ by the following composition:

$$
\begin{aligned}
&F'(F^{n-1}End(X)) \times \Delta^n \xrightarrow{1 \times (s_{n-1})^{n-1} \times s_0} \\
&\to F'(F^{n-1}End(X)) \times \Delta^2 \times \Delta^{n-1} \xrightarrow{k_{n-1} \times 1} \\
&\to F^{n-1}End(X) \times \Delta^{n-1} \xrightarrow{h_{n-1}} End(Y).
\end{aligned}
$$

On the summand I we define

$$I \times \Delta^n \xrightarrow{1 \times (s_{n-1})^{n-1}} I \times \Delta^1 \xrightarrow{\varepsilon \times \psi_1} \underline{K}_{\mathscr{S}}(Y,Y) \times End_1(Y) \to End_1(Y).$$

It is straightforward to verify, using the inductive description of $F^*(\mathscr{E})$ given in Proposition 3.1, that the morphisms h_n, $n \geq 0$ satisfy the necessary coherent conditions.

For a construction of a coherent isomorphism ϕ see [17, Theorem 6.1(iii)]. To make excuses to the reader, note that we do not use anywhere in our paper the second part of the theorem. $\square$

5. Cosimplicial $\mathscr{E}$-algebras and $\mathscr{E}$-coalgebras

Let A be a category with finite colimits and let $X^{*,*}$ be a bicosimplicial object in A. Then we can construct a cosimplicial object $\nabla(X) = \nabla(X^{*,*})$ as follows [2]:

$$\nabla(X)^0 = X^{0,0}.$$

$\nabla(X)^{n+1}$ is the colimit of the diagram

$$\begin{tikzcd} \dots \arrow[r] & X^{p,q+1} & \\ & X^{p,q} \arrow[u, "d_2^0"] \arrow[r, "d_1^{p+1}"] & X^{p+1,q} \\ & & \arrow[u] \quad \dots \end{tikzcd}$$

$p+q=n.$

See also [2] for the definition of cofaces and codegeneracies.

Remark. A construction of this type was used for the first time in [1]. By this reason ∇ is called in [12] the Artin–Mazur codiagonal functor. As is noted in [12], $\nabla(X)$ is none other than the left Kan extension along the ordinal sum functor.

Let now K be a monoidal $\mathscr{A}$-category with finite colimits. Then we can define the tensor product $\otimes_{cK}$ of cosimplicial objects in K by the formula

$$X^* \otimes_{cK} Y^* = \nabla((X \hat{\otimes}_K Y)^{*,*}),$$

where $(X\hat{\otimes}_K Y)^{*,*}$ is a bicosimplicial object of the type

$$(X\hat{\otimes}_K Y)^{p,q} = X^p \otimes_K Y^q$$

with the obvious cofaces and codegeneracies [2].

Proposition 5.1. *If $\otimes_K$ commutes with colimits, then the category cK of cosimplicial objects of K is a monoidal $\mathscr{A}$-category with respect to the tensor product defined above and with the constant cosimplicial object $I^n = I_K$ as the unit object. Furthermore, if K is (strongly) $\mathscr{A}$-tensored then cK is also (strongly) $\mathscr{A}$-tensored.*

Proof. Immediate from the definitions. □

Let Δ^n be the standard topological simplex

$$\Delta^n = \{(u_1,\dots,u_n) \in \Re^n \mid 0 \le u_1 \le \dots \le u_n \le 1\}.$$

Let $\Delta[s]$, $0 \le s \le \infty$ be the cosimplicial space , consisting of the s-skeleton of Δ^n with usual cofaces and codegeneracies between them. We denote $\Delta[\infty]$ by Δ.

Theorem 5.1. *Let H be the operad defined in Section 3. Then for each $0 \le s \le \infty$ the cosimplicial space $\Delta[s]$ has an H-coalgebra structure in $c\mathscr{T}op$. For all $s_0 \le s_1$ the natural inclusions $\Delta[s_0] \subset \Delta[s_1]$ are the morphisms of H-coalgebras.*

Proof. We shall prove this theorem for $s=\infty$, the other part of the theorem will follow immediately.

To obtain a coaction of H on Δ we have to construct the mappings

$$\gamma_{m,n} : \Delta^m \times H_n \to (\underbrace{\Delta \otimes \cdots \otimes \Delta}_{n})_m, \quad m, n \geq 0,$$

where we denote by $\otimes$ the tensor product in $c\mathcal{T}op$. At the beginning we want to describe some algorithm, which associates with every vertex A of H_n some subdivision of the unit interval $[0,1]$.

Recall that a vertex A of H_n is a word consisting of n symbols $a_1, \ldots, a_n$, l symbols 1 and some pairs of brackets. We call a subword of A a subsequence of A, which is either one of the symbols $a_i, 1$ or is the expression taken in brackets in A. The set of the subwords of A has a natural partial ordering. Namely, $B_0 > B_1$ if B_1 is a subword of B_0. Let $p_1 = 1$ if A consists only of one symbol, otherwise let p_1 be the number of the maximal elements of the poset of proper (not equal to A) subwords of A. Consider the subdivision of $[0,1]$ on p_1 intervals $[(j-1)/n, j/n]$, $j = 1, p_1$. We associate the interval $[(j-1)/n, j/n]$ with the jth maximal subword of A. We can continue this process of subdivision by applying the same algorithm to the interval $[(j-1)/n, j/n]$ and the set of proper subwords of A_j for each j. Finally, we obtain a subdivision of $[0,1]$ on $n+l$ intervals, such that with every symbol a_i, $i = 1, n$ and every symbol 1 contained in A its own interval is associated. If the symbol has the number j in A (in the natural order) then we denote the corresponding interval by $[\sigma_{j-1}, \sigma_j]$, $0 \leq j \leq n+l$, $\sigma_0 = 0, \sigma_{n+l} = 1$.

For example, let A be a word $(11(a_1a_2)1)a_3$. Then the corresponding subdivision of $[0,1]$ is

$$[0,1/8],\ [1/8,1/4],\ [1/4,5/16],\ [5/16,3/8],\ [3/8,1/2],\ [1/2,1].$$

A subdivision described allows us to define the following subdivision of Δ^n on the subspaces $P^{r_1,\ldots,r_{n+l}}$, $\sum_{i=1}^{n+l} r_i = m$:

$$P^{r_1,\ldots,r_{n+l}} = \{u \in \Delta^m | \sigma_0 \leq u_1 \leq \cdots \leq u_{r_1} \leq \sigma_1 \leq u_{r_1+1} \\ \leq \cdots \leq u_{r_1+r_2} \leq \sigma_2 \leq \cdots \leq u_m \leq 1\}.$$

Let now $[\sigma_{j_k-1}, \sigma_{j_k}]$ be an interval associated with symbol a_k, $k = 1, n$. Define a map

$$\gamma_{m,n} : \Delta^m \to (\underbrace{\Delta \otimes \cdots \otimes \Delta}_{n})_m$$

on the vertex A by the formula

$$\gamma_{m,n}(u) = b(\alpha^1(u), \ldots, \alpha^n(u)), \quad u \in P^{r_1,\ldots,r_{n+l}},$$

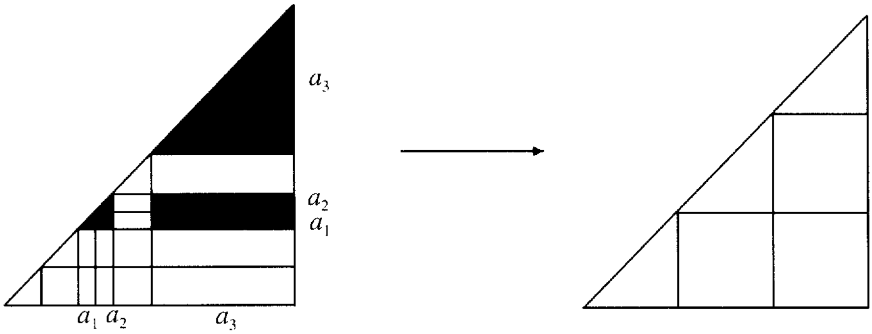
where b is the canonical morphism in colimit and

$$\alpha^k : P^{r_1,\ldots,r_{n+l}} \to \Delta^{r_{j_k}}$$

is defined by

$$\alpha^k(u) = \left(\frac{u_{r_1+\cdots+r_{j_k-1}+1} - \sigma_{j_k-1}}{\sigma_{j_k} - \sigma_{j_k-1}}, \ldots, \frac{u_{r_1+\cdots+r_{j_k}} - \sigma_{j_k-1}}{\sigma_{j_k} - \sigma_{j_k-1}} \right).$$

By way of illustration we present $\gamma_{2,3}$, corresponding to the vertex A from the above example, in the following picture:



Let now

$$A_0 \longleftarrow \cdots \longleftarrow A_k$$

be a k-simplex of H_n. Then we can construct the $k+1$ subdivisions of $[0,1]$ on $n+l$ intervals by applying the process above to each A_i and assuming that we insert a trivial interval $[\sigma,\sigma]$ in place of the symbols 1 thrown off. Let us demonstrate this on an example.

Consider the following 2-simplex of H_3:

$$(11(a_1a_2)1)a_3 \longleftarrow ((a_1a_2)1)a_3 \longleftarrow a_1a_2a_3.$$

Then we obtain 3 subdivisions of $[0,1]$ on 6 intervals:

$$[0,1/8],\ [1/8,1/4],\ [1/4,5/16],\ [5/16,3/8],\ [3/8,1/2],\ [1/2,1].$$

$$[0,0],\ [0,0],\ [0,1/8],\ [1/8,1/4],\ [1/4,1/2],\ [1/2,1].$$

$$[0,0],\ [0,0],\ [0,1/3],\ [1/3,2/3],\ [2/3,2/3],\ [2/3,1].$$

Return to the general construction. Let

$$[\sigma_{j-1}(s),\sigma_j(s)],\quad 1 \le j \le n+l,\ 0 \le s \le k$$

be the interval corresponding to a symbol with number j in the sth subdivision. Let $t=(t_0,\ldots,t_k) \in \Delta^k$ and let

$$\Theta_i(t) = t_{i+1} - t_i,\quad t_0 = 0,\quad t_{k+1} = 1,\quad 0 \le i \le k.$$

Then define

$$P^{r_1,\dots,r_{n+l}}(t) = \left\{ (u,t) \in \Delta^m \times \Delta^k \,\middle|\, 0 \le u_1 \le \cdots \le u_{r_1} \le \sum_{s=0}^{k} \Theta_s(t)\sigma_1(s) \right.$$
$$\left. \le u_{r_1+1} \le \cdots \le u_{r_1+r_2} \le \sum_{s=0}^{k} \Theta_s(t)\sigma_2(s) \le \cdots \le u_m \le 1 \right\}.$$

Then we put $\gamma_{m,n}$ on the simplex $A_0 \leftarrow \cdots \leftarrow A_k$ of H_n equal to

$$\gamma_{m,n}(u,t) = b(\alpha^1(u,t),\dots,\alpha^n(u,t)), \quad (u,t) \in P^{r_1,\dots,r_{n+l}}(t),$$

where $\alpha^i(u,t)$ is defined by the same formulas as $\alpha^i(u)$ but instead of σ_{j_i} we use $\sum_{s=0}^{k} \Theta_s(t)\sigma_{j_i}(s)$.

It is straightforward to check that we thus obtain a coaction of H on Δ. □

Remark. Our subdivision $P^{r_1,\dots,r_{n+l}}(t)$ is an immediate generalization of subdivision of Δ^n introduced in [18, 8, 9, 2]. For example, the comultiplication

$$\rho : \Delta \to \Delta \otimes \Delta$$

is the image under γ of the vertex $a_1 a_2$ of H_2 [2].

Corollary 5.1.1. *Consider $\mathcal{T}op$ as an $\mathcal{S}$-category. Then the cosimplicial topological space $\Delta[s]$, $0 \le s \le \infty$ has an $\mathcal{H}$-coalgebra structure.*

Let now K be a complete monoidal $\mathcal{A}$-category. Then for any cosimplicial objects M^* of K and E^* of $\mathcal{A}$ we define the realization of M^* with respect to E^* by

$$(M^*)^{E^*} = \int_p (M^p)^{E^p}.$$

Theorem 5.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two $\mathcal{A}$-operads. If $\otimes_K$ commutes with finite colimits then for any $\mathcal{E}_1$-algebra M^* in cK and $\mathcal{E}_2$-coalgebra E^* in $c\mathcal{A}$ the realization, $(M^*)^{E^*}$, is and $(\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2)$-algebra in K.*

Furthermore, this construction is functorial, that is we have the realization as an $\mathcal{A}$-functor

$$(Coalg^{\mathcal{E}_2}(c\mathcal{A})) \otimes_{\mathcal{A}} (Alg^{\mathcal{E}_1}(cK)) \to Alg^{\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2}(K).$$

Proof. Let us show firstly that there exists an $\mathcal{A}$-natural transformation

$$\tau : (N^*)^{E^*} \otimes_K (M^*)^{F^*} \to (N^* \otimes_{cK} M^*)^{E^* \otimes_{c\mathcal{A}} F^*},$$

which makes monoidal the functor of realization.

Indeed, we have an obvious transformation

$$\tau_0 : (N^*)^{E^*} \otimes_K (M^*)^{F^*} = \int_p (N^p)^{E^p} \otimes_K \int_q (M^q)^{F^q}$$
$$\to \int_{p,q} (N^p)^{E^p} \otimes_K (M^q)^{F^q} \to \int_{p,q} (N^p \otimes_K M^q)^{E^p \otimes_{\mathscr{A}} F^q}.$$

Let now X be an object of K, then

$$\underline{K}\left(X, \int_{p,q} (N^p \otimes_K M^q)^{E^p \otimes_{\mathscr{A}} F^q}\right) \simeq \int_{p,q} \underline{K}(X, (N^p \otimes_K M^q)^{E^p \otimes_{\mathscr{A}} F^q})$$
$$\simeq \int_{p,q} \underline{\mathscr{A}}(E^p \otimes_{\mathscr{A}} F^q, \underline{K}(X, N^p \otimes_K M^q)) \xrightarrow{\nabla} \underline{c\mathscr{A}}(\nabla((E \,\hat{\otimes}_{\mathscr{A}} F)^{*,*}),$$

$$\nabla(\underline{K}(X, (N \hat{\otimes}_K M)^{*,*}))) \to \underline{c\mathscr{A}}(\nabla((E \,\hat{\otimes}_{\mathscr{A}} F)^{*,*}), \underline{K}(X, \nabla((N \hat{\otimes}_K M)^{*,*})))$$
$$= \underline{c\mathscr{A}}(E^* \otimes_{c\mathscr{A}} F^*, \underline{K}(X, N^* \otimes_{cK} M^*))$$
$$= \int_n \underline{\mathscr{A}}((E^* \otimes_{c\mathscr{A}} F^*)^n, \underline{K}(X, (N^* \otimes_{cK} M^*)^n)) \simeq \underline{K}(X, (N^* \otimes_{cK} M^*)^{E^* \otimes_{c\mathscr{A}} F^*}).$$

Thus by the enriched Yoneda lemma [16] we have a morphism

$$\tau_1 : \int_{p,q} (N^p \otimes_K M^q)^{E^p \otimes_{\mathscr{A}} F^q} \to (N^* \otimes_{cK} M^*)^{E^* \otimes_{c\mathscr{A}} F^*}.$$

Then we put $\tau = \tau_1 \cdot \tau_0$. It is not hard to check that τ has the desired properties.

Let m be the following $\mathscr{A}$-natural transformation:

$$\underline{cK}(M^*, N^*) \otimes_{\mathscr{A}} \underline{c\mathscr{A}}(E^*, F^*) \to \underline{K}((M^*)^{F^*}, (N^*)^{F^*}) \otimes_{\mathscr{A}} \underline{K}((N^*)^{F^*}, (N^*)^{E^*})$$
$$\xrightarrow{\mu} \underline{K}((M^*)^{F^*}, (N^*)^{E^*}),$$

where μ is the composition in K. Using the transformations m and τ we obtain

$$\underline{cK}((M^*)^j, N^*) \otimes_{\mathscr{A}} \underline{c\mathscr{A}}(E^*, (F^*)^j) \xrightarrow{m} \underline{K}(((M^*)^j)^{(F^*)^j}, (N^*)^{E^*})$$
$$\xrightarrow{\tau} \underline{K}(((M^*)^{F^*})^j, (N^*)^{E^*}).$$

This provides us with a natural transformation

$$r : \overline{\underline{cK}}(M^*, N^*) \otimes_{\mathscr{A}} \overline{\underline{c\mathscr{A}}}^o(E^*, F^*) \to \overline{\underline{K}}((M^*)^{F^*}, (N^*)^{E^*}).$$

Finally, using the $\mathscr{E}_1$-algebra structure on M^* and the $\mathscr{E}_2$-coalgebra structure on E^*, we obtain a morphism of $\mathscr{A}$-collections

$$\mathscr{E}_1 \otimes_{\mathscr{A}} \mathscr{E}_2 \to End(M^*) \otimes_{\mathscr{A}} End^o(E^*) \xrightarrow{r} End((M^*)^{E^*}).$$

It is straightforward to verify that this morphism is a morphism of operads. The naturality of the constructions involved gives us the second part of the theorem. □

Corollary 5.2.1. *Let $\mathscr{A}$ be a structure of a model Quillen category. If in $\mathscr{A}$ the tensor product of weak equivalences is a weak equivalence, then under the conditions of Theorem 5.2 the realization of an A_∞-monoid in cK with respect to an A_∞-comonoid in $c\mathscr{A}$ is an A_∞-monoid in K.*

6. Some applications to coherent homotopy theory

The results obtained permit us to generalize the construction of the coherent homotopy category of a comonad (a monad) introduced in [2]. Let K be an $\mathscr{A}$-category and let (L, ρ, ε) be an $\mathscr{A}$-comonad on K. Let $\mathscr{E}$ be an $\mathscr{A}$-operad and let N be an $\mathscr{E}$-coalgebra in $c\mathscr{A}$. Then we can consider an $\mathscr{A}$-endodistributor on K,

$$\underline{K}(L_*(X), Y)^N = (\phi^{L_*})^N(X, Y)$$

(see Section 2 for the definition of a simplicial $\mathscr{A}$-functor L_*).

Proposition 6.1. *The distributor*

$$(\phi^{L_*})^N$$

has a natural structure of an $\mathscr{E}$-algebra in Dist(K, K).

If, in addition, K is cocomplete, then the endofunctor

$$L_* \otimes_{\mathscr{A}} N(X) = \int^p L^{p+1}(X) \otimes_{\mathscr{A}} N^p$$

has a natural $\mathscr{E}$-coalgebra structure in F(K, K).

Proof. We can apply Theorem 5.2 if we prove that the cosimplicial $\mathscr{A}$-distributor ϕ^{L_*} has a structure of a monoid in *Dist*(K, K). Define a multiplication as follows:

$$\begin{aligned}
\mu^{pq} : \underline{K}(L^{p+1}(X), Y) \otimes_{\mathscr{A}} \underline{K}(L^{q+1}(Y), Z) &\to \\
\underline{K}(L^{q+1}L^{p+1}(X), L^{q+1}(Y)) \otimes_{\mathscr{A}} \underline{K}(L^{q-1}(Y), Z) &\to \\
\underline{K}(L^{q+1}L^{p+1}(X), Z) &\xrightarrow{L^q \cdot \rho \cdot L^p} \underline{K}(L^{q+p+1}(X), Z).
\end{aligned}$$

The counit ε generates obviously some cosimplicial morphism $I \to \phi^{L_*}$. It is easy to check that we thus obtain a needed monoid structure (see also [2] for the dual case of a monad).

The second part of the proposition follows from Proposition 1.4. $\square$

To relate this construction with that of coherent homotopy category we give the following

Definition 6.1. We call the homotopy category of an A_∞-graph G in $\mathscr{A}$ the category with objects those of G and with $\pi_0 \underline{\mathscr{A}}_{\mathscr{S}}(I_{\mathscr{A}}, G(X, Y))$ as the morphisms from X to Y.

The composition and identity morphisms are induced by the A_∞-monoid structure in the obvious way.

The proposition just established and Theorem 5.1 give

Theorem 6.1. *Let (L, ρ, ε) be a $\mathscr{T}op$-comonad on a $\mathscr{T}op$-category K. For each $0 \leq s \leq \infty$ the distributor $(\phi^{L_*})^{\Delta[s]}$ has a natural structure of H-algebra in $Dist(K,K)$. A natural morphism of distributors*

$$I \to (\phi^{L_*})^{\Delta[s]},$$

generated by ε, is the morphism of H-algebras.

If K is cocomplete, then the endofunctor $L_* \times \Delta[s]$ has a natural H-coalgebra structure in $F(K,K)$.

The coherent homotopy category of (L, ρ, ε) with respect to $\Delta[s]$ is the homotopy category of A_∞-graph $((\phi^{L_*})^d)^{\Delta[s]}$.

Let now K be a cocomplete $\mathscr{T}op$-category and let (R, μ, η) be a $\mathscr{T}op$-monad on K. Let, in addition, R commute with the functor of geometric realization. Then we have

Corollary 6.1.1. *May's bar-construction* [20, 21]

$$B(R,R,X) = \int_p B_p(R,R,X) \times \Delta^p$$

is the H-coalgebra in the category of $\mathscr{T}op$-endofunctor on the category of R-algebras.

The coherent homotopy category of R-algebras is the homotopy category of an A_∞-graph, which assigns to R-algebras X and Y a topological space

$$\underline{Alg}^R(B_*(R,R,X), Y).$$

The simplicial analogue of the above theory is of special interest. But we cannot develop it immediately, because the cosimplicial simplicial set Δ has no A_∞-comonoid structure. This is the same kind of difficulty which was considered in [2]. To overcome it we use the construction from example 4, Section 1.

Let C be a simplicial category, then one can consider the category of topological distributors $Dist_{\mathscr{T}op}(C, C)$ considering $\mathscr{T}op$ as a simplicial category. The fibrewise singular complex functor provides us some monoidal (not strong) $\mathscr{S}$-functor

$$S : Dist_{\mathscr{T}op}(C, C) \to Dist(C, C).$$

Proposition 6.2. *Let (T^*, μ^*, η^*) be a monoid in $cDist_{\mathscr{T}op}(C, C)$. Then a simplicial endodistributor $Tot(S(T^*))$ has a natural structure of $\mathscr{H}$-algebra in $Dist(C, C)$. Moreover, a morphism*

$$\eta_\infty : I \to Tot(S(T^*))$$

induced by unit η^* is a morphism of $\mathscr{H}$-algebras.

Proof. Note that the singular complex functor maps the H-algebras in $Dist_{\mathcal{T}op}(C,C)$ to the $\mathscr{H}$-algebras in $Dist(C,C)$. This follows from the monoidality of S and the fact that the natural morphism $\mathscr{H} \to S(H)$ is a morphism of $\mathscr{S}$-operads.

Let X, Y be objects of C, then

$$Tot(S(T^*))(X,Y) = \int_n \underline{\mathscr{L}}(\Delta(n), S(T^n(X,Y)))$$

$$\simeq \int_n \underline{\mathcal{T}op_{\mathscr{S}}}(\Delta^n, T^n(X,Y)) = S((T^*)^\Delta(X,Y)).$$

But $(T^*)^\Delta$ is an H-algebra and hence $Tot(S(T^*)) \simeq S((T^*)^\Delta)$ is an $\mathscr{H}$-algebra. $\square$

Remark. The proposition just proved is very useful in categorical strong shape theory developed in [3].

We are now ready to prove a simplicial analogue of Theorem 6.1.

Theorem 6.2. *Let (L,ρ,ε) be an $\mathscr{S}$-comonad on an $\mathscr{S}$-category K. If*

$$\underline{K}(L_*(X), Y)$$

is a fibrant cosimplicial simplicial set for every $X, Y \in ob(K)$ then the locally Kan graph $Tot(\phi^{L_*})^d$ has a natural structure of H-algebra and the coherent homotopy category $CHL_\infty - K$ is isomorphic to the homotopy category of this graph.

Proof. From the conditions of the theorem we have a homotopy equivalence of Kan simplicial sets

$$Tot(\phi^{L_*})(X,Y) \sim Tot(S(|\phi^{L_*}(X,Y)|))$$

where $|-|$ is fibrewise (with respect to cosimplicial structure) geometric realization. The topological cosimplicial distributor $|\phi^{L_*}|$ is a cosimplicial monoid. Then by Proposition 6.2 $Tot(S(|\phi^{L_*}|))$ has a structure of $\mathscr{H}$-algebra. Now by Theorems 3.1 and 4.1 the locally Kan graph $Tot(\phi^{L_*})^d$ has a structure of an $F_\infty(\mathscr{M})$-algebra and hence of $\mathscr{H}$-algebra. $\square$

Corollary 6.2.1. *Let K be a locally Kan monoidal $\mathscr{S}$-category with strong $\mathscr{S}$-tensorization and let $\mathscr{E}$ be an $\mathscr{S}$-operad.*

Then the coherent homotopy category of $\mathscr{E}$-algebras is a homotopy category of the locally Kan A_∞-graph $Tot(\phi^{[\mathscr{E}]_*})^d$.

Finally, we can apply Theorem 6.2 to show that the coherent homotopy category of the diagrams may be obtained by a natural way as a homotopy category of some locally Kan A_∞-graph.

Recall now some definitions of coherent homotopy theory developed in [2, 8–10, 12].

The following construction is extracted from [12]. Let A be a small $\mathcal{S}$-category. For objects X, Y of A form the bisimplicial set $\Psi(X,Y)$ defined by

$$\Psi(X,Y)_{n,*} = \coprod_{A_0,\dots,A_n} \underline{A}(X,A_0) \times \underline{A}(A_0,A_1) \times \cdots \times \underline{A}(A_n,Y)$$

where

$$d_i : \Psi(X,Y)_{n,*} \to \Psi(X,Y)_{n-1,*}$$

is defined by composition in A,

$$\underline{A}(A_{i-1},A_i) \times \underline{A}(A_i,A_{i+1}) \to \underline{A}(A_{i-1},A_{i+1}),$$

and $s_i : \Psi(X,Y)_{n,*} \to \Psi(X,Y)_{n+1,*}$ by the canonical morphism

$$\Delta^0 \to \underline{A}(A_i,A_i).$$

Set now $\check{A}(X,Y) = Diag(\Psi(X,Y))$.

Definition 6.2. Let B be a complete $\mathcal{S}$-category, with cotensorization

$$(-)^{(-)} : B \times \mathcal{S}^{op} \to B,$$

and let

$$T : A^{op} \times A \to B$$

be an $\mathcal{S}$-functor.

The simplicially coherent end of T will be the object $\oint_A T(A,A)$ of B defined by

$$\oint_A T(A,A) = \int_{A^{op}\times A} T(A',A'')^{\check{A}(A',A'')}.$$

As was established in [12], for the coherent end the cosimplicial replacement formula and a universal property like that of homotopy limit are fulfilled.

Let Λ be a small simplicial category, and let $F, G : \Lambda \to K$ be two simplicial functors to an $\mathcal{S}$-category K. Then we can consider as the simplicial set of coherent natural transformations from F to G the coherent end [8, 10, 12]:

$$\underline{Coh}(F,G) = \oint_\Lambda \underline{K}(F(\lambda), G(\lambda)).$$

As was shown in [12], for weakly locally Kan category K one can define a composition of coherent transformations, which is associative up to coherent homotopy. On the level of the set of homotopy classes of coherent transformations we thus get the category $coh(\Lambda,K)$ called the coherent homotopy category of the diagrams of the type Λ.

Theorem 6.3. *If K is cocomplete locally Kan $\mathcal{S}$-category then the assignment*

$$F, G \mapsto \underline{Coh}(F,G)$$

has the structure of a locally Kan simplicial A_∞-graph. The coherent homotopy category of the diagrams coh(Λ, K) is the homotopy category of this graph.

Proof. Consider the discretization Λ_d of the category Λ and the inclusion $i : \Lambda_d \to \Lambda$. Denote $F(\Lambda, K)$ the $\mathscr{S}$-category of $\mathscr{S}$-functors from Λ to K. Then i gives rise to a pair of $\mathscr{S}$-adjoint functors [16]

$$i^* : F(\Lambda,K) \to F(\Lambda_d,K), \ Lan_i : F(\Lambda_d,K) \to F(\Lambda,K),$$

and thus we have some $\mathscr{S}$-comonad (L,ρ,ε) on $F(\Lambda,K)$.

The proof consists now in verification that

$$\underline{Coh}(F,G) \simeq Tot(\phi^{L_*})(F,G),$$

which is an immediate corollary of the cosimplicial replacement formula [12]. Furthermore, as K is locally Kan the cosimplicial simplicial set $\phi^{L_*}(F,G)$ is fibrant [12, 2] and so we are in the conditions of Theorem 6.2. □

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